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Revisiting the proof theory of Classical S4

Revisitando a teoria da prova de S4

Abstract

In 1965 Dag Prawitz presented an extension of Gentzen-type systems of Natural Deduction to modal concepts of S4. Maria da Paz Medeiros showed in 2006 that the proof of normalisation for classical S4 does not hold and proposed a new proof of normalisation for a logically equivalent system, the system NS4. However two problems in the proof of the critical lemma used by Medeiros in her proof were pointed out by Yuuki Andou in 2009. This paper presents a proof of the critical lemma, resulting in a proof of normalisation for NS4.

Keywords: modal logic; proof theory; normalisation.

Resumo

Em 1965, Dag Prawitz apresentou uma extensão dos sistemas tipo-Gentzen de Dedução Natural para os conceitos modais de ***S4****. Maria da Paz Medeiros mostrou em 2006 que a prova de normalização para o* ***S4*** *clássico não estava correta e propôs uma nova prova de normalização para um sistema logicamente equivalente, o sistema* ***NS4****. No entanto, dois problemas na prova do lema crítico usado por Medeiros em sua prova foram apontados por Yuuki Andou em 2009. Este artigo apresenta uma nova prova do lema crítico e, consequentemente, uma prova de normalização para* ***NS4****.*

Palavras-chave: lógica modal; teoria da prova; normalização.

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1. Introduction

In his Ph.D. thesis, Dag Prawitz (2006) extended the Gentzen-type systems of Natural Deduction (ND) to modal concepts, obtaining Gentzen-type systems of ND for S4 based on classical, intuitionistic and minimal predicate logic. For this purpose, a modal operator of necessity (here represented by $\Box$) was added together with the rules for its introduction and elimination. Prawitz then presented three formalizations of those modal systems, which differed in the restrictions on the introduction rule for $\Box$, and stated that only the third one would accept the Normalisation Theorem.

About forty years later, Maria da Paz Medeiros (2006) argued that Prawitz's normalisation procedure does not hold on the third version of the ND system for classical S4, and proposed a new system for S4, the system NS4, for which the Normalisation Theorem would hold.

However, recently Yuuki Andou (2009a) pointed out two problems in the proof of a lemma (namely the critical lemma) that plays a crucial role in Medeiros' proof of the Normalisation Theorem. Andou (2009a) presented a Normal Form Theorem, showing that for any proof Π there is a proof Π' in normal form by means of cut-elimination, but do not present a normalisation procedure.

In this paper we present a correction of Medeiros' proof of the aforementioned lemma and fulfil a normalisation procedure for NS4, which gives a computational interpretation of proofs by means of the Curry-Howard Correspondence. Despite the fact that the existence of a normalised version of any proof of S4 is already proved, to the best of our knowledge this is the first proof of normalisation for which a normalisation procedure is presented for a Natural Deduction system for S4.

After some definitions used in the present work (Section2), we outline the original third version of Prawitz's system for classical S4 and the counterexamples by Medeiros (Section3). We then discuss the two cases in which the system may not produce valid derivations on NS4 due to problems in the proof of the critical lemma (Section 4). In Section 5 we present a proof of the critical lemma for NS4. Our concern here is with Classical Propositional S4, but an extension of Classical First Order S4 could be easily obtained by adding the corresponding rules for quantifiers.

2. Definitions

Based in the work of Maria da Paz Medeiros (2006), we present the definitions used in this work.

Definition 1

The premisses $(A\rightarrow B)$ of the rule $(E\rightarrow)$,$(A\wedge B)$ of, $(E\wedge)$,$(A\vee B)$ of $(E\vee)$,$(\Box A)$ of $(E\Box)$, and the premisses $\Box B_1,\ldots,\Box B_n$ of $(I\Box)$ are called major premisses and the others minor premisses.

Definition 2

A segment in a derivation is a sequence $A_1,\ldots,A_n$ of occurrences of the same formula in a branch such that A_1 is not the conclusion of an application of $(E\vee)$nor a discharged assumption through an application of $(I\Box)$, and A_n is not a minor premiss of $(E\vee)$ nor a major premiss of $(I\Box)$.

Definition 3

The length of a segment is the number of formula occurrences in this segment.

Definition 4

A maximal segment in a derivation is a segment $A_1,\ldots,A_n$such that A_1 is the conclusion of an application of an introduction rule or $(\bot c)$, and A_n is a major premiss of an application of an elimination rule.

Definition 5

A maximal formula is a maximal segment whose length is 1(one). A premiss is called a maximal premiss if it belongs to some maximal segment.

Definition 6

A formula A is a trivial formula if A is the conclusion of an application of ($\perp$c) and the minor premiss of an application of($E\rightarrow$) whose major premiss is the assumption $\neg A$.

Definition 7

The degree of a formula A, G(A), is the number of occurrences in A of logical symbols different from $\perp$. The degree of a segment is the degree of the formula that belongs to this segment.

Definition 8

The degree of a derivationΠ, $G(\Pi)$ is the highest degree of a maximal segment of Π. If Π does not have maximal segments, then $G(\Pi)=0$.

Definition 9

A critical derivation is a derivation Π such that, if $G(\Pi)=d$, then the last inference of Π has a maximal premiss with degree d, and for every subderivation Σ of Π, $G(\Sigma)<G(\Pi)$.

Definition 10

The index of a derivation Π is $I(\Pi)=\langle d, s\rangle$, where s is the sum of the lengths of the maximal segments of Π whose degree is d. If Π does not have maximal segments, then $I(\Pi)= \langle 0, 0\rangle$.

Definition 11

A derivation Π is a normal derivation if Π does not have maximal segments.

Definition 12

A formula A is essentially modal when each occurrence of a predicate parameter or predicate constant in A stands within the scope of an occurrence of $\Box$.

3. A counterexample for Prawitz's classical S4 system

According to the restriction on the □-I rule in Prawitz's third version of S4, if a formula A depends on an assumption B and there exists an essentially modal formula F on the thread of A from B such that A depends on every assumption which F depends on, then □-I could be applied at A.

But Maria da Paz Medeiros (2006) argued that such restriction would not avoid maximal formulas by pointing out that although the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\Box A]^1 \qquad \neg\Box A \to B}{B} \qquad \neg B}{\bot}}{\Box A}\,1}{A}}{C \to A}}{\Box(C \to A)}$$

is valid in **S4**, its reduction is not:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box A]^1}{A} \qquad [\neg A]^2}{\bot}\,1}{\neg\Box A} \qquad \neg\Box A \to B}{B} \qquad \neg B}{\bot}}{A}\,2}{C \to A}}{\Box(C \to A)}}{}$$

It was presented a new Natural Deduction for **S4** system, called system **NS4** and proposed a normalisation proof by critical lemma Medeiros (2006). This new system is composed of the logical symbols ∧,∨,→,⊥ and □ and the usual rules, except for □-I, which is as follows:

$$\dfrac{\Box\vec{B} \qquad \begin{array}{c}[\Box\vec{B}]^k\\ \Lambda\\ A\end{array}}{\Box A}\,k$$

(We use $\begin{array}{c}\vec{F}\\ \Box\vec{B}\end{array}$ for a sequence of deductions of the form $\begin{array}{c}\Pi_i\\ \Box B_i\end{array}$ where no two □Bi's are equal. We also write[□ $\vec{B}$]k to indicate that each □Bi is discharged at k. Labels may be dropped.)

This restriction on □-I rule states that all the assumptions in [□ $\vec{B}$]k must be discharged by the application of □-I and the premiss A must not depend on any assumption other than the ones in □ $\vec{B}$. The reason for this is explained in item 4 of the proof of the critical lemma and it does not affect the completeness of the system.

Together with this new □-I rule, we have the following reduction:

$$\dfrac{\dfrac{\begin{array}{c}\vec{\Sigma}\\ \Box\vec{B}\end{array} \qquad \begin{array}{c}[\Box\vec{B}]^k\\ \Lambda\\ A\end{array}}{\Box A}\,k}{A} \quad \rhd \quad \begin{array}{c}\vec{\Sigma}\\ (\Box\vec{B})\\ \Lambda\\ A\end{array}$$

(1)

and the permutative reduction bellow:

$$\dfrac{\dfrac{\begin{array}{c}\vec{\Sigma}\\ \Box\vec{A}\end{array} \qquad \begin{array}{c}[\Box\vec{A}]^i\\ \Lambda_1\\ B\end{array}}{\Box B}\,i \qquad \begin{array}{c}\vec{\Psi}\\ \Box\vec{D}\end{array} \qquad \begin{array}{c}[\Box B]^j[\Box\vec{D}]^k\\ \Lambda_2\\ C\end{array}}{\Box C}\,k,j \quad \rhd$$

$$\dfrac{\begin{array}{c}\vec{\Sigma}\\ \Box\vec{A}\end{array} \qquad \begin{array}{c}\vec{\Psi}\\ \Box\vec{D}\end{array} \qquad \begin{array}{c}\dfrac{[\Box\vec{A}]^j \qquad \begin{array}{c}[\Box\vec{A}]^k\\ \Lambda_1\\ \mathrm{B}\end{array}}{(\Box B)}\,k \qquad [\Box\vec{D}]^i\\ \Lambda_2\\ \mathrm{C}\end{array}}{\Box C}\,j,i$$

(2)

4. A problem in the normalisation proof of NS4

Medeiros' normalisation proof begins with the assumption that a derivation Π of C from Γ can be transformed in a derivation Π0. The aim is to show that I(Π0) <I(Π). Next, it uses a critical lemma according to which, if Π is a critical derivation of C from Γ, then Π can be transformed into a derivation Π' such that I(Π')<I(Π). By the critical lemma, a subderivation Σ of Π can be transformed in a subderivation Σ' such that I(Σ')<I(Σ); but, then, Π1 is the derivation resulted from the substitution of Σ' for Σ in Π0, and I(Π1) <I(Π0).

However, recently Yuuki Andou (2009b) pointed out two flaws in the proof of Medeiros' critical lemma. The first one concerns critical derivations of the form

$$\Pi \equiv \dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\Sigma_{0,1}\\ F\end{array}\quad [\neg F]^i}{\bot}\\ \Sigma_{0,2}\\ \bot\end{array}}{F}\,i \qquad \begin{array}{c}\vec{\Sigma}\\ \vec{H}\end{array}}{C}\,r \tag{3}$$

where the major premiss F is the conclusion of⊥c and r is an elimination rule. According to Medeiros, this derivation can be transformed into

$$\Pi' \equiv \begin{array}{c}\dfrac{\begin{array}{c}\Sigma_{0,1}\\ F\end{array}\quad \begin{array}{c}\vec{\Sigma}\\ \vec{H}\end{array}}{\bot}\,r\\ \Sigma_{0,2}\\ \bot\end{array} \quad \text{or to} \quad \Pi'' \equiv \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma_{0,1}\\ F\end{array}\quad \begin{array}{c}\vec{\Sigma}\\ \vec{H}\end{array}}{C}\,r \qquad [\neg C]^i}{\bot}\\ \Sigma_{0,2}\\ \bot\end{array}}{C}\,i \tag{4}$$

depending on C being ⊥ or not.

Note that the assumptions of the form ¬F discharged at the rule i may occur more than once in Π, and that the premiss F which is conclusion of Σ0,1 may be a maximal premiss in Π' and in Π''. In this case the index of either Π' or Π'' may be even greater than that of Π. Besides, one of the Hi's in $\vec{H}$, say Hl, may be a maximal formula of degree G(F) and, in this case, even if the F side connected with ¬F is not a maximal formula in Π', this Hl still is and the induction hypothesis cannot be used.

The second problem pointed out is when Π has degree G(□A) and is a critical derivation of the form:

$$\Pi \equiv \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vec{\Sigma}\\ \Box\vec{B}\end{array} \quad \begin{array}{c}[\Box\vec{B}]^k\\ \Lambda_1\\ A\end{array}}{\Box A}\,k \quad [\neg\Box A]^i}{\bot}\\ \Sigma_0\\ \bot\end{array}}{\Box A}\,i \quad \begin{array}{c}\vec{\Psi}\\ \Box\vec{C}\end{array} \quad \begin{array}{c}[\Box A]^j[\Box C]^\ell\\ \Lambda_2\\ B\end{array}}{\Box B}\,\ell, j \tag{5}$$

If □A occurs more than once as top-formula of Λ2, by reducing Π to

$$\Pi' \equiv \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vec{\Sigma}\\ \Box\vec{B}\end{array} \quad \begin{array}{c}\vec{\Psi}\\ \Box\vec{C}\end{array} \quad \begin{array}{c}\dfrac{[\Box\vec{B}]^j \quad \begin{array}{c}[\Box\vec{B}]^k\\ \Lambda_1\\ A\end{array}}{\Box A}\,k \quad [\Box C]^\ell\\ \Lambda_2\\ B\end{array}}{\Box B}\,\ell, j \quad [\neg\Box B]^i}{\bot}\\ \Sigma_0\\ \bot\end{array}}{\Box B}\,i$$

the number of occurrences of □A as maximal formula in Π' will be greater than in Π.

Thus, it is possible that the reduction process generates copies of maximal formulas, so the index of Π' may be greater than that of Π.

5. Yet another proof of the critical lemma

We present a proof of the critical lemma for the fragment $\{\wedge, \rightarrow, \perp, \Box\}$. Extensions to First Order Logic are easily obtained.

Lemma 1. A critical derivation of the form

$$\Pi \equiv \dfrac{\dfrac{\begin{matrix}[\neg F]^1 \\ \Sigma_1 \\ \perp\end{matrix}}{F}\,1 \qquad \begin{matrix}\vec{\Sigma} \\ \vec{B}\end{matrix}}{C}\,r$$

where F is the conclusion of $\perp$c, can be transformed in a derivation

$$\Pi_1 \equiv \dfrac{\dfrac{\begin{matrix}\dfrac{\begin{matrix}\Sigma'_{1,1} \\ F\end{matrix} \qquad [\neg F]^1}{\perp} \\ \Sigma'_{1,2} \\ \perp\end{matrix}}{F}\,1 \qquad \begin{matrix}\vec{\Sigma} \\ \vec{B}\end{matrix}}{C}\,r$$

WhereΠ1 is a derivation without trivial formulas. Thus, the end-formula of Σ′1,1 is not conclusion of ⊥c.

Proof. See the work of Medeiros (2006).

Note that Π1 has no more maximal formulas of degree equal to or higher than G(F) than Π. We will use the symbol $\propto$to indicate the transformation of a derivation into a derivation without trivial formulas.

Theorem 5.1 If Π is a critical derivation of C from Γ, then Π can be transformed into a derivation Π' such that G(Π')<G(Π).

Suppose Π is a critical derivation with maximal premisses of degree G(Π) which are premisses of the last inference of Π, #G(Π) is the number of maximal formulas of Π with degree G(Π) and l(Π) is the lenght of Π. The proof is by induction on the pair ⟨#G(Π), l(Π)⟩.

$$1.\ \Pi \equiv \dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ A\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ B\end{matrix}}{A \wedge B}}{A} \quad \rhd \quad \begin{matrix}\Sigma_1 \\ A\end{matrix} \equiv \Pi'$$

It is easy to see that G(Π′) <G(Π).

$$2.\ \Pi \equiv \dfrac{\begin{matrix}\Sigma_1\\ A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \Sigma_2\\ B\end{matrix}}{A \to B}}{B} \quad \rhd \quad \begin{matrix}\Sigma_1\\ (A)\\ \Sigma_2\\ B\end{matrix} \quad \equiv \Pi'$$

It is easy to see that $G(\Pi') < G(\Pi)$.

$$3.\ \Pi \equiv \dfrac{\dfrac{\begin{matrix}\vec{\Sigma}\\ \Box\vec{B}\end{matrix} \qquad \begin{matrix}[\Box\vec{B}]\\ \Lambda_1\\ C\end{matrix}}{\Box C}}{C} \quad \rhd \quad \begin{matrix}\vec{\Sigma}\\ (\Box\vec{B})\\ \Lambda_1\\ C\end{matrix} \quad \equiv \Pi'$$

If there exists a $\Box Bl$ which is a maximal premiss at Π', then it would be a maximal formula at Π and, as Π is a critical derivation, $G(\Box Bl) < G(\Box C)$. Thus, $G(\Pi') < G(\Pi)$.

$$4.\ \Pi \equiv \dfrac{\dfrac{\begin{matrix}\vec{\Sigma}\\ \Box\vec{B}\end{matrix} \qquad \begin{matrix}[\Box\vec{B}]^k\\ \Lambda_1\\ A\end{matrix}}{\Box A}{\scriptstyle k} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}[\Box A]^j, [\vec{H}]^l\\ \Lambda_2\\ C\end{matrix}}{\Box C}{\scriptstyle j,l} \quad \rhd$$

$$\dfrac{\begin{matrix}\vec{\Sigma}\\ \Box\vec{B}\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}\dfrac{[\Box\vec{B}]^l \qquad \begin{matrix}[\Box\vec{B}]^k\\ \Lambda_1\\ A\end{matrix}}{(\Box A)}{\scriptstyle k} \qquad [\vec{H}]^j\\ \Lambda_2\\ C\end{matrix}}{\Box C}{\scriptstyle j,l} \quad \equiv \Pi_1$$

We have two cases to consider:

(a) there is an occurrence of $\Box A$ which is top-formula of Λ_2 and major premiss of an application of $\Box$-E: in this case, the number of maximal formulas of degree $G(\Box A)$ in Π_1 may be even greater than that of Π, as there may be more than one occurrence of $\Box A$ as top-formula of Λ_2.

There is a critical subderivation Ξ_1 of Π_1 of the form $\dfrac{\dfrac{\Box\vec{B} \qquad \begin{matrix}[\Box\vec{B}]^k\\ \Lambda_3\\ A\end{matrix}}{\Box A}{\scriptstyle k}}{A}$ which can be reduced to $\Xi_1' \equiv \begin{matrix}\Box\vec{B}\\ \Lambda_3\\ A\end{matrix}$ (case 3)

(b) there is an occurrence of □A which is top-formula of Λ2 and major premiss of □-I : then, there is a critical subderivation Ξ2 of the form

$$\dfrac{\dfrac{\Box\vec{B} \qquad \begin{matrix}[\Box\vec{B}]^k\\ \Lambda_3\\ A\end{matrix}}{\Box A}\,k \qquad \begin{matrix}\vec{\Upsilon}\\ \Box\vec{C}\end{matrix} \qquad \begin{matrix}[\Box A]^p, [\Box\vec{C}]^q\\ \Lambda_4\\ D\end{matrix}}{\Box D}\,p,q$$

The lenght of Ξ2 is smaller than the lenght of Π. Hence, by the induction hypothesis, we can reduce Ξ2 to a Ξ′2 such that G(Ξ′2) <G(Π).

Note that we cannot guarantee that the lenght of Ξ2 is smaller than the lenght of Π if there were more than one occurrence of □A as top-formula of Λ2 in Π1, and if there were many occurrences of □A as major premiss in Ξ2. That is the reason of the restriction on the beginning of the section.

Let Π2 be the result of replacing each occurrence of critical subderivations of the form of Ξ1 and the form of Ξ2 in Π1 by Ξ′1 and Ξ′2 respectively.

If □A is the only major premiss that is maximal formula in Π, i.e., there is no member of $\vec{H}$ which is a maximal premiss of the same degree of Π, then G(Π2) <G(Π) and Π2 = Π. Otherwise, i.e., if there exists a m such that Hm is a maximal formula in Π, then #G(Π2) < #G(Π) and, as Π2 is a critical derivation, by the induction hypothesis Π2 can be transformed into a derivation Π′ such thatG(Π′) <G(Π2). Hence, as Π was transformed into Π2 and G(Π2) is not higher than G(Π), G(Π′) <G(Π).

$$5.\ \Pi \equiv \dfrac{\dfrac{\begin{matrix}[\neg(A\wedge B)]^1\\ \Sigma_1\\ \bot\end{matrix}}{A\wedge B}\,1}{A} \quad \propto \quad \dfrac{\dfrac{\begin{matrix}\dfrac{\begin{matrix}\Sigma'_{1,1}\\ A\wedge B\end{matrix} \qquad [\neg(A\wedge B)]^1}{\bot}\\ \Sigma'_{1,2}\\ \bot\end{matrix}}{A\wedge B}\,1}{A} \quad \rhd \quad \dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\Sigma'_{1,1}\\ A\wedge B\end{matrix}}{A} \qquad [\neg A]^1}{\bot}\\ \Sigma'_{1,2}\\ \bot\end{matrix}}{A}\,1 \quad \equiv \Pi_1$$

If the end formula of Σ′1,1 is not the conclusion of an introduction rule, then the end-formula of Σ′1,1 is not a maximal formula and G(Π1) <G(Π) and Π1 ≡ Π′. If the end formula of Σ′1,1 is the conclusion of an introduction rule, then Π1 is of the form

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}\Sigma_3\\A\end{array}\quad\begin{array}{c}\Sigma_4\\B\end{array}}{A\wedge B}}{A}\qquad[\neg A]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{A}\,1\end{array}$$

which can be reduced to

$$\begin{array}{c}\dfrac{\begin{array}{c}\Sigma_3\\A\end{array}\qquad[\neg A]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{A}\,1\end{array}\equiv\Pi'$$

and G(Π′) < G(Π).

6. $$\Pi\equiv\dfrac{\dfrac{\begin{array}{c}[\neg(A\to B)]\\\Sigma_1\\\bot\end{array}}{A\to B}\qquad\begin{array}{c}\Sigma_2\\A\end{array}}{B}\ \propto\ \dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\Sigma'_{1,1}\\A\to B\end{array}\qquad[\neg(A\to B)]^1}{\bot}\\\Sigma'_{1,2}\\\bot\end{array}}{A\to B}\,1\qquad\begin{array}{c}\Sigma_2\\A\end{array}}{B}\ \rhd$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma'_{1,1}\\A\to B\end{array}\quad\begin{array}{c}\Sigma_2\\A\end{array}}{B}\qquad[\neg B]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{B}\,1\end{array}\equiv\Pi_1$$

If the end formula of Σ′1,1 is not the conclusion of an introduction rule, then G(Π1) <G(Π) and Π1 ≡ Π′. If the end formula of Σ′1,1 is the conclusion of an introduction rule, then Π1 is of the form:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\\Sigma_3\\B\end{array}}{A\to B}\quad\begin{array}{c}\Sigma_4\\A\end{array}}{B}\qquad[\neg B]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{B}\,1\end{array}$$

which can be reduced to

$$\begin{array}{c}\dfrac{\begin{array}{c}\Sigma_4\\(A)\\\Sigma_3\\B\end{array}\qquad[\neg B]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{B}\,1\end{array}\equiv\Pi'$$

and G(Π′) < G(Π).

7. $$\Pi\equiv\dfrac{\dfrac{\begin{array}{c}[\neg\Box A]\\\Sigma_1\\\bot\end{array}}{\Box A}}{A}\ \propto\ \dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\Sigma'_{1,1}\\\Box A\end{array}\qquad[\neg\Box A]^1}{\bot}\\\Sigma'_{1,2}\\\bot\end{array}}{\Box A}\,1}{A}\ \rhd\ \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Sigma'_{1,1}\\\Box A\end{array}}{A}\qquad[\neg A]^1}{\bot}\\\Sigma'_{1,2}\\\dfrac{\bot}{A}\,1\end{array}\equiv\Pi_1$$

If the end formula of Σ′1,1 is not the conclusion of an introduction rule, then G(Π1) <G(Π) and Π1 ≡ Π′. If the end formula of Σ′1,1 is the conclusion of an introduction rule, then Π1 is of the form:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{matrix}\vec{\Psi}\\ \Box\vec{B}\end{matrix} \quad \begin{matrix}[\Box\vec{B}]^k\\ \Lambda_1\\ A\end{matrix}}{\Box A}\,{\scriptstyle k}}{A} \qquad [\neg A]^l}{\perp}\\ \Sigma'_{1,2}\\ \dfrac{\perp}{A}\,{\scriptstyle k,l}\end{array}$$

which can be reduced to

$$\begin{array}{c}\dfrac{\begin{matrix}\vec{\Psi}\\ (\Box\vec{B})\\ \Lambda_1\\ A\end{matrix} \qquad [\neg A]^l}{\perp}\\ \Sigma'_{1,2}\\ \dfrac{\perp}{A}\,{\scriptstyle l}\end{array} \equiv \Pi_2$$

If one of the Bi’s, say Bm, were a maximal formula in Π2, it would be a maximal formula in Π and, as Π is a critical derivation, G(Bm) <G(A). Thus, G(Π2) <G(Π) and Π2 ≡ Π′.

$$8.\ \Pi \equiv \dfrac{\dfrac{\begin{matrix}[\neg\Box A]^k\\ \Sigma_1\\ \perp\end{matrix}}{\Box A}\,{\scriptstyle k} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}[\Box A]^l, [\vec{H}]^j\\ \Sigma_2\\ C\end{matrix}}{\Box C}\,{\scriptstyle l,j} \quad \propto$$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{matrix}\Sigma'_{1,1}\\ \Box A\end{matrix} \qquad [\neg\Box A]^k}{(\perp)}\\ \Sigma'_{1,2}\\ \perp\end{array}}{\Box A}\,{\scriptstyle k} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}[\Box A]^l, [\vec{H}]^j\\ \Sigma_2\\ C\end{matrix}}{\Box C}\,{\scriptstyle l,j} \quad \rhd$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{matrix}\Sigma'_{1,1}\\ \Box A\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}[\Box A]^l, [\vec{H}]^j\\ \Sigma_2\\ C\end{matrix}}{\Box C}\,{\scriptstyle j,l} \qquad [\neg\Box C]^k}{(\perp)}\\ \Sigma'_{1,2}\\ \dfrac{\perp}{\Box C}\,{\scriptstyle k}\end{array} \equiv \Pi_1$$

Note that Σ′1,1 is a subderivation of Σ1. Hence, if the subderivation

$$\Lambda \equiv \dfrac{\begin{matrix}\Sigma'_{1,1}\\ \Box A\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix} \qquad \begin{matrix}[\Box A]^l, [\vec{H}]^j\\ \Sigma_2\\ C\end{matrix}}{\Box C}\, l, j$$

of Π1 is a critical derivation, its lenght is smaller than the lenght of Π. Thus, by the induction hypothesis, Λ can be reduced to a derivation Λ′ such that G(Λ′) <G(Π). The result of replacing each occurrence of Λ in Π1 by Λ′ is a derivation Π′ such that G(Π′) <G(Π).

$$9.\ \Pi \equiv \dfrac{\begin{matrix}[\neg F]^1\\ \Sigma_1\\ \dfrac{\bot}{F}\,1\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix}}{\bot}\, r \quad \propto \quad \dfrac{\begin{matrix}\dfrac{\begin{matrix}\Sigma'_{1,1}\\ F\end{matrix} \qquad [\neg F]^1}{\bot}\\ \Sigma'_{1,2}\\ \dfrac{\bot}{F}\,1\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix}}{\bot}\, r \quad \rhd\ \Pi_1 \equiv \begin{matrix}\dfrac{\begin{matrix}\Sigma'_{1,1}\\ F\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix}}{\bot}\\ \Sigma'_{1,2}\\ \bot\end{matrix}$$

$$\Lambda \equiv \dfrac{\begin{matrix}\Sigma'_{1,1}\\ F\end{matrix} \qquad \begin{matrix}\vec{\Psi}\\ \vec{H}\end{matrix}}{\bot}$$

The critical subderivation of Π1 is smaller than Π. Thus, by the induction hypothesis, Λ can be reduced to a derivation Λ′ such that G(Λ′) <G(Π). By replacing each occurrence of Λ in Π1 by Λ′ we achieve the desired derivation. This case deals with classical ⊥with the elimination of implication, conjunction and box.

$$10.\ \Pi \equiv \begin{matrix}\dfrac{[\neg F]^1 \qquad \begin{matrix}\Sigma'\\ \neg F \to C\end{matrix}}{C}\\ \Sigma\\ \dfrac{\bot}{F}\,1\\ \overline{\ B\ }\,\text{el}\end{matrix} \quad \rhd \Pi' \equiv \begin{matrix}\dfrac{\dfrac{\dfrac{\dfrac{[F]^1}{B} \qquad [\neg B]^2}{\bot}}{\neg F}\,1 \qquad \begin{matrix}\Sigma'\\ \neg F \to C\end{matrix}}{C}\\ \Sigma\\ \dfrac{\bot}{B}\,2\end{matrix}$$

If ¬F → C were a maximal formula in Π′, it would be a maximal formula in Π, which is not the case as, by hypothesis, G(Π) = G(F) and G(F) <G(¬F → C). Hence, G(Π) <G(Π′).

6. Conclusions

This work presented the problem pointed out by Maria da Paz Medeiros (2006) on the normalisation procedure proposed by Dag Prawitz (2006), followed by the system suggested by Medeiros (2006), the NS4. She presented a normalisation proof for this system for which we presented the problems pointed out by Yuuki Andou (2009b) and finally we presented a proof of the Normalisation Theorem for NS4.

Among other deductive systems for **S4**, there are some where the Normalisation Theorem holds, like Sequent Calculus. There is also a Natural Deduction with Labels system by Alex Simpson Simpson (1994) for which the Normalisation Theorem holds. But the system proposed by Dag Prawitz (2006) and Maria da Paz Medeiros (2006) are pure Natural Deduction systems, without semantic interferences (as the labels from the system of Alex Simpson) for which there are no previous proof of the Normalisation Theorem known into the available literature. Yuuki Andou showed that for any proof of **S4** there is a normalised proof via cut-elimination Andou (2009a) but did not present a normalisation procedure. We fulfilled that gap by presenting a correction in Medeiros' proof that lead to a normalised Natural Deduction system for **S4**, the **NS4** system.

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