Some Remarks on Wittgenstein's Philosophy of Mathematics: The Case of Euler's Proof

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Resumo

Neste artigo discuto as críticas feitas por Wittgenstein, na seção 123 do Big Typescript, a uma demonstração dada por Leonard Euler para a infinitude dos números primos. Wittgenstein propõe ainda uma correção, que é o único exemplo conhecido de trabalho matemático realizado pelo autor. Nas considerações tecidas por Wittgenstein ao redor desse tema, encontram-se algumas das questões centrais de sua filosofia da matemática, tal como desenvolvida a partir do período intermediário. Meu objetivo é separar os diferentes aspectos da crítica wittgensteiniana, ao mesmo tempo em que sugiro um modo de interpretá-la que faça jus a outra posição que Wittgenstein nunca se cansa de repetir: A matemática não tem nada a temer da filosofia, pois a filosofia não pode levar à rejeição de nenhum cálculo matemático. Termino por examinar quais seriam, de acordo com as concepções de Wittgenstein, as verdadeiras tarefas da filosofia em relação à matemática.

Palavras-Chaves: Filosofia da matemática em Wittgenstein . Big Typescript . Prova de Euler . Construtivismo em matemática

Abstract

In the present paper I discuss the criticisms levelled by Wittgenstein in Big Typescript, section 123 at Leonard Euler's proof of the infinity of prime numbers. Wittgenstein also offers a correction of that proof, which is his only known piece of mathematical work. Among the issues at stake here are some of the most central to his philosophy of mathematics, as developed from the middle period on. My purpose is to disentangle the different aspects of his criticisms and suggest a way of interpreting them that does justice to another claim he never ceases to repeat: mathematics has nothing to fear from philosophy, for philosophy cannot lead to the rejection of any mathematical calculus. I conclude by examining what would be, on Wittgenstein's view, the real tasks of philosophy with respect to mathematics.

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In *Big Typescript*, section 123 Wittgenstein discusses a proof given by Leonard Euler establishing the infinity of prime numbers. In a recent paper[1], M. Marion and P. Mancosu analyze this section – as well as a number of points relative to its historical context –, with an acute commentary and many interesting results. It is only fair to say that I am very indebted to the authors, and acknowledge that their research provided the main spark for the reflections I will develop in the following.

As a matter of fact, around the end of their paper, Marion & Mancosu write: "We are now left with the task of reconstructing this *Standpunkt Wittgensteins* and the above discussion of his remarks on the constructivization of Euler's proof should be seen as a contribution to this task". I took the recommendation seriously, and I would like the discussion below to be taken in the very same spirit: as an attempt to further our understanding of Wittgenstein's standpoint with regard to the philosophy of mathematics.

First of all, let me try to summarise some of the points made by Marion & Mancosu in their paper:

1) Section 123 of the *Big Typescript* ends with some awkward calculations, the purpose of which is very difficult to figure out. Marion & Mancosu give the manuscript sources (MS 108) from which the calculations are drawn, and show that they contain a complete version of these calculations, making it possible to fully understand them.

2) Based on this finding, they show that Wittgenstein's calculations actually amount to a *correction* of Euler's proof, a proof taken by Wittgenstein to be inadmissible in its original form.

3) Supported by the now complete calculations, interpreted as a correction of Euler's proof, and by a number of remarks available in Wittgenstein's text, they argue for the following position: Wittgenstein is not content merely with purifying the discourse *about* mathematics; he has something to say about the way mathematics – and mathematical proofs – are conducted.

1 [Marion & Mancosu: 2003]

It is with respect to this last claim that I will try to improve upon. I believe some of the issues at stake here are utterly important for a correct understanding of the *Standpunkt Wittgensteins*. Let us then take a look at Euler, Wittgenstein, and Wittgenstein about Euler.

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The proof of the infinitude of the prime numbers given by L. Euler in the last decades of the 18th century – a proof I will discuss in detail in the next section – makes use of the "reductio ad absurdum". Its general structure is as follows.

It begins with an equation, taken to be valid[2]:

$$\sum_{n \in \mathbb{N}} \frac{1}{n} = \prod_{p \text{ is prime}} \frac{1}{1 - {}^{1}/_{p}}$$

The left side of this equation is an infinite sum (more precisely: it is a divergent series, known as the "harmonic series"). The right side is a product, ranging over the prime numbers. Now, if there were a finite amount of prime numbers, the right side would be finite. But that is impossible, by virtue of its being equated to the left side, which is infinite. Therefore, the prime numbers cannot be finite – they are infinite.

With respect to this proof, Wittgenstein expresses himself, indeed, in a very critical manner. He puts the situation in his usual colourful way:

> *"(...) Here once again we have that remarkable phenomenon that we might call circumstantial proof in mathematics – <u>something that is absolutely never permitted</u>. Or perhaps we could call it a proof by symptoms. The result of the summation is (or is understood as) a symptom that there are terms on the left that are missing on the right. The connection of the symptom to what we would like to have proved is loose. That is, no bridge has been built, so we settle for seeing the other bank.*
> *All the terms on the right side occur on the left, but the sum on the left side yields ∞ and the one on the right only a finite value – so . . . must; <u>but in</u>*

2 The reasons why this equation is taken to be valid will not concern us here. Suffice it to say that there is a fairly unanimous agreement upon its validity, from both the point of view of traditional and constructive mathematics.

> *mathematics the only thing that must be is what is.*
> *The bridge has to be built."* [3] (My underlines)

Wittgenstein tries to make us see that there is something very problematic in the way Euler's proof is presented. But what is, exactly, the target of his criticism? Here the complications begin. The examination of section 123 of the *Big Typescript* suggests various candidates, whose mutual connections are far from simple. There are at least four of them:

I. In the first paragraph of the above-quoted passage, Wittgenstein seems to criticize a *method of demonstration* which he labels "proof by circumstantial evidence" (*Indizienbeweis)* or "proof by symptoms" (*Beweis durch Symptome*).

II. Closely related to the rejection of this method of demonstration is, as it seems, the rejection of the "law of excluded middle" as a general logical law. The law is explicitly mentioned, by the end of the section, in the discussion of an example[4].

III. In the second paragraph of the passage quoted above, there are some critical remarks regarding the use of modality (the use of "must") in mathematics. Interpretation here becomes particularly tricky.

IV. Finally, we can find an attack directed at the relation between Euler's demonstration and that which the proof purports to be a demonstration of: *"The connection of the symptom to what we would like to have proved is loose"*.

All these are possible targets of Wittgenstein's criticism, and they maintain complex relations among themselves. I do not doubt the different themes can be adequately related to one another, assuming their place in a coherent interpretation of Wittgenstein's point of view. In what follows, however, I will not examine all these possibilities, nor discuss their mutual relations. I shall follow a different strategy.

As it happens, there is in Wittgenstein's text what I take to be a particularly enlightening formulation of the main problems he is dealing with. The formulation I have in mind is what I shall call the "criticism of the bet". After presenting Euler's proof, Wittgenstein asks:

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3 Big Typescript, section 123 (p. 434e)

4 "What follows from that? (The law of excluded middle.) Nothing follows from that, except that the limiting values of the sums are different; that is, nothing [new]." BT, section 123 (p. 435e)

> *"(...) If one had only this proof, what would one stake on it? If, say, we had found the primes up to N, could we then go on infinitely in search of a further prime number – since the proof gives us the guarantee that we will find one? Surely that is nonsense. – For 'if we only search long enough' means nothing at all."*[5] (My underline)

Such criticism is not only deep. It follows in the clearest manner from the very central issues about mathematics Wittgenstein has been trying to establish since section 108, the first of the mathematical sections of the book.

Put in a nutshell, the point of view Wittgenstein is trying to force on us as absolutely inescapable is this: Mathematics is not a descriptive science. It has no special domain of idealized objects to describe. Neither is it an empirical science. It is *just* a calculus: a set of prescriptions for dealing with symbols. In particular, being purely normative, it does not require any justification from the empirical world and, what is perhaps more difficult to accept, it cannot hope to find in the empirical world any parameters for its correction. Hard as it may be, mathematics finds nothing in the empirical world which to lean upon. It is *just* a set of rules to manipulate symbols. If we search to prop it up against something other than these rules, we are astray.

Let us take a look, then, at what I called, in connection with Euler's proof, the "criticism of the bet". Euler's proof purportedly guarantees us that, given any prime number, there is a bigger one (because the supposition of a last prime number ends up in contradiction). If P is the biggest prime number known today, Euler's proof guarantees us that there is another prime number bigger than P. But what does "guarantee" mean here? As a piece of mathematics, it cannot consist in some kind of subjective confidence. It cannot be an act of faith.

What Wittgenstein wants to face us with, at this point, is something like the dialogue between two disagreeing mathematicians. The sceptical mathematician protests: "I am not content with Euler's proof, and I am not convinced there is a prime number bigger than P". The traditional mathematician answers: "But the proof has shown us there *must* be one". The sceptic replies: "The proof *shows* nothing; it may confront us with an alleged contradiction, but it shows us nothing remotely like a prime number bigger than P. Anyway, what does it mean to say there *must* be a prime number bigger than P?

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5 BT, section 123 (p. 435e)

What does it mean to say that the proof *guarantees* its existence? What is the guarantee?" [6].

The traditional mathematician finds himself, suddenly, in a difficult position. What could he reply to these objections? Being an honest mathematician, he cannot avail himself of a platonic realm of numbers, where this prime number bigger than P is somehow supposed *to be* (but which nobody knows where to find). So our mathematician tries the other way round and, being very confident in the proof, declares himself willing to bet all his money on its correction: "Just give me *sufficient time* and I will find you a prime number bigger than P, as Euler's proof tells us there must be one".

But how long is "sufficient time"? After a year of unsuccessful search by his adversary, can the sceptic mathematician claim the money? The traditional mathematician would deny: "Just give me some *more time*; Euler's proof guarantees *there is* a prime number bigger than P, not that I would find it within a year; this number is somewhere, I know; I just need to search *long enough*".

The problem has now clearly arisen. This bet, the only bet Euler's proof can lead us into, does not possess the *mathematical meaning* it should have. The act of "searching long enough" can certainly have some meaning, but only an empirical one. The expression "long enough" is liable to many empirical specifications – a day, a year, five hundred centuries –, but as such it has nothing to do with mathematics. It is a mathematical specification we are lacking here. Thus, we see that it is from the mathematical point of view that the act of searching long enough "means nothing at all", as Wittgenstein puts

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6 At this point, the traditional mathematician could try to pursue something like the following line of argument: "The proof shows us that <u>if</u> mathematics is to be a consistent set of propositions, <u>then</u> there must be no biggest prime number". There wouldn't be much gain in it, however. The sceptic would observe that the introduction of the conditional form, as well as the talk about a "set of (mathematical) propositions", only introduces further mystery and confusion in the argument, but does not touch, in the least, the central difficulty. Here is the reply: "You talk about consistency of a set of propositions, and you adopt a conditional form, believing to have made some progress. But the issue continues to be very much the same: What does it mean to say that if (...), then there <u>must</u> be a prime number bigger than P? The problem, which lies in the consequent, remains untouched. 'If mathematics is to be consistent', then Euler's proof guarantees the existence... But what is the guarantee?"

The main source of difficulty – as we will see later in this paper – rests on an insufficient understanding of the relation between the different calculi which constitute mathematics. What we call the consistency of a certain calculus (e. g. the calculus used to formulate Euler's proof) may not have to the system of natural numbers the relation we expect. As Wittgenstein puts it, the connection is loose ("of the symptom to what we would like to have proved").

it. As a mathematical result, Euler's demonstration has no value. But that is the only value it could have[7].

The argumentation is irreproachable. Euler's demonstration has, as far as this line of considerations goes, no mathematical value. But if we have come all the way up here, then we have to ask: What, exactly, is devoid of mathematical value in Euler's demonstration – a demonstration taken to be good by almost any mathematician? What goes wrong with it?

It is my opinion that the first thing to do is to isolate – literally to isolate – the core problem denounced by Wittgenstein's criticism. What is this core problem? I believe that what Wittgenstein's approach leads us to reject, primarily and directly, is the acceptability of the *proposition* crowning Euler's proof: "Given any prime number P, there is another prime number bigger than P". It is this *proposition* which has "absolutely no meaning", as shown by the "criticism of the bet". It has no meaning because, taken at face value, it tries to describe what cannot be described. In fact, the only meaning we could attach to it is as the description of a realm of mathematical facts. The expression "there is", as used there, can only be associated with a description, and not with an activity of symbolic manipulation. But this is precisely the big mistake Wittgenstein never ceases to draw our attention to: mathematics is solely an activity of manipulating symbols; it describes nothing.

Up to this point, I take Wittgenstein's criticism to be as sound as it can be. However, if the proposition crowning Euler's proof is unacceptable – at least if taken literally –, what to say about the proof itself?

I mentioned above that Wittgenstein's criticism, variously expressed, seems to be directed at the proof proper. I actually tried to pinpoint four possible targets of his criticism. Moreover, we have the additional clues offered by the work of Marion & Mancosu. Having traced the unfinished calculations of the *Big Typescript* back to their manuscript sources, wherein Wittgenstein brings them to completion, they argue convincingly that Wittgenstein actually intended to correct Euler's proof.

7 As we may now see, Wittgenstein is playing with two different but equally naive beliefs mathematicians use to profess (at least in moments of despair). They oscillate between taking for granted some platonic realm of mathematical objects and facts, whose description would be the purpose of mathematics, and nourishing some vague hope that mathematics is liable, after all, to empirical confirmation (since we can use the mathematical calculus – so the idea goes – to "make predictions"). In other words: when a mathematician recognizes that the platonic supposition is an inadequate account of what he is doing, he frequently tries to find shelter in the empirical world – failing to notice, once again, that empirical prediction is not what he is doing (is not what mathematics is about).

But what kind of correction is it?

Wittgenstein's purpose is to determine a precise number-interval where one can be sure to find a prime number bigger than another prime number already available. Given such an interval, the situation clears up completely. If we have the prime number P, the reworked proof establishes there is another prime number bigger than P *and smaller than some value P'*. This is now a perfect mathematical proposition. It tells us something very definite about the manipulation of symbols. And now we have a bet with completely clear mathematical meaning (which is, by the way, a very stupid bet for someone to make; it can only be conceived in a situation of ignorance or lack of understanding of the mathematical proof).

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Now I propose to take a careful look at Euler's original proof – something which is very much in line with Wittgenstein's general stance toward such cases.

Its main logical steps are as follows:

(a) The initial equality between the harmonic series and certain product taken over the primes:

$$\sum_{n \in \mathbb{N}} \frac{1}{n} = \prod_{p \text{ is prime}} \frac{1}{1 - {}^{1}/_{p}}$$

(b) Divergence of the harmonic series (the harmonic series grows bigger than any specified number).

(c) If there is a greatest prime number, the product on the right is finite.

(d) The harmonic series on the left, being divergent, would grow bigger than this finite product.

(e) Therefore, the product cannot be finite.

(f) Therefore, there must be an infinite stock of prime numbers.

The two questionable passages of the proof seem to be (e) and (f). But before considering them, it is worth examining in more detail what goes on at (b), (d) and (c). (We may assume equality (a) to be valid, as already observed.)

As regards (b): The divergence of the harmonic series is obtained by way of a *demonstration*. How is it done? By showing that, given any value M, it is possible to specify another value s(M) such that, if the series is continued up to s(M), it will become bigger than M. This value s(M) is perfectly determined.

As regards (d): The impossibility of equating a divergent summation with a finite product has, in accordance, the following structure: If the product on the right is any finite number M, then there is a value s(M) such that the summation on the left, getting to s(M), grows bigger than M. This is the reason *why the equality is discarded, for any number M.*

As regards(c): If there is only a finite stock of prime numbers, there is one prime number P which is the biggest of them all. The product will then stop at the term relative to P, yielding as result certain number M(P).

We now put (c) and (d) together: Because the harmonic series diverges, there is a number s(M(P)) such that the summation, getting to s(M(P)), grows bigger than M(P).

And here we see the important point: It suffices to take the harmonic series beyond s(M(P)) to be sure to find some prime number bigger than P. And *that* concludes the proof.

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We saw that Wittgenstein was very much at odds with Euler's proof, which he calls "circumstantial proof in mathematics" (or "proof by symptoms") and condemns beyond redemption: "something that is absolutely never permitted".

As I argued above, the primary problem with Euler's proof was that its final assertion is inadmissible. A final assertion that can be put in different forms – there are infinite primes; there is no prime greater than all others; given any prime P, there is another prime greater than P –, all of them destitute of mathematical meaning. Or, to use the "argument of the bet": all of them leading to unacceptable, non-mathematical bets.

Wittgenstein's correction of the proof – what Marion & Mancosu call his "constructivization" of the proof – amounts to performing some additional calculations. These calculations have the effect that, at the end of the reworked proof, we obtain an assertion of the following form: Given any prime number P, there is a prime number greater than P *and smaller than some value P'*. In other words, the calculations exhibited by Wittgenstein establish a range of search for the new prime number whose existence is asserted.

However, as I tried to show above, it is exactly this that Euler's proof *does*. Given any finite product, it shows that this product will be overtaken by the harmonic series at some point. As a matter of fact, Euler's proof is to be understood within the framework of a demonstration[8] of the divergence of the harmonic series, which can be used to determine the exact point where the overtaking is bound to occur, for any prime number P – the point I dubbed s(M(P)).

To put it in the crudest way: Euler's proof is, on a careful examination, perfectly constructive.

But now we seem to face this strange situation: a perfectly sound proof, ending in an inadmissible proposition. What then is going on?

What happens is that Euler's proof does not incorporate, as part of its final assertion, the very analysis that is contained in it (an analysis which alone enabled the proof to be worked out). If the analysis had been incorporated in the final result, the final result would read something like this: "Given any prime number P, there is another prime number bigger than P *and smaller than the denominator corresponding to s(M(P))*". This would be, by Wittgenstein's own criteria, a perfectly acceptable mathematical assertion; and it is contained in Euler's proof, which is therefore perfectly acceptable according to these same criteria.

Euler judges it unnecessary to encumber the final result with all the cargo of the analysis contained in the demonstration. He expresses it simply as "the number of primes is infinite" or "given any prime P, there is another prime bigger than P". It is chiefly a matter of expediency[9]. All the vocabulary used there has this same character: abbreviating an otherwise tiresome repetition of all the conditions involved in the assertions.

Is this a condemnable option? Yes and no.

As an abbreviation, it cannot be condemned. However, it is a dangerous abbreviation. It does treat the situation as if it were a case of describing some ethereal mathematical world, where the numbers really are, in all their infinitude; and it does lead the mathematician to confusion about what he is actually doing. To put it as Wittgenstein does: the mathematician loses sensitivity about the nature of his work[10].

I said "abbreviation"; maybe I should have used the term "omission". When this omission occurs – when Euler drops out from the final assertion the sound

8 A trivially constructive demonstration.

9 Usual mathematical language has a lot to gain in simplicity by the use of abbreviated forms such as "the sum is infinite", without the burden of making it explicit, at every moment, all the details contained in such assertions ("given any value M, it suffices to go on with the sum until the term s(M) in order for the sum to reach a value greater than M").

10 BT, section 122 (p.432e)

mathematical structure of his proof –, what ensues is an assertion that, taken literally, is destitute of meaning. An assertion that seems to be describing what is not there to be described, and that points to a mathematically unacceptable bet. And this, I believe, is what Wittgenstein would like us to see. To quote two phrases used by him when discussing these issues, Euler's result "surrounds mathematics with a mist"[11] and with "false interpretations"[12].

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Marion & Mancosu call Wittgenstein's work in manuscript 108 (a work only partially to be seen in section 123 of the *Big Typescript*) a "constructivization" of Euler's Proof. I tried to show in what sense we should understand this constructivization: what is actually amiss in the original proof (and what is *not* amiss); what is the gain in Wittgenstein's new formulation; what is at stake in this issue.

We saw that Euler's original proof already had, when looked at carefully, a constructive structure. More precisely: that the necessary constructive determinations are already at work in the original proof. We saw, however, that this constructive structure is kept hidden or implicit, and is not incorporated to the final statement of the proof. We then examined why Wittgenstein considered this an intolerable situation (from the philosophical point of view certainly, but with consequences also for the work of professional mathematicians), and the pains he took to correct it.

To bring this discussion to an end, I would like to focus on the last problem, the most important from the point of view of understanding Wittgenstein's work in the philosophy of mathematics. What are exactly Wittgenstein's efforts and worries when dealing with a result such as Euler's? What should we take his real concern to be? Here we have two options. They are closely related to one another, but they are not quite the same. It is worth disentangling them.

Firstly, we may take Wittgenstein's major concern to be that of correcting the mathematical result – a result that is to be regarded as wrong, at least to some extent (to what extent?). This kind of interpretation is suggested by the use of a term such as "constructivization", which conveys the idea of transforming a (wrong) non-constructive proof into a (correct) constructive one.

11 BT, section 120 (p. 424e)

12 BT, section 123 (p. 435e)

There follows a very distinctive approach: Wittgenstein's position would be that of "correcting" actual mathematics, and somehow rejecting parts of it – something he denied repeatedly.

I believe this is not the best option, for reasons I will expand on below. However, I must point out that such an approach has much to be said in its favour, especially in the light of the foregoing discussions. Wittgenstein condemns Euler's "circumstantial proof" as something "absolutely never permitted"; and we saw what strong reasons he had to consider the statement ensuing from Euler's proof as destitute of mathematical meaning. If this is not rejecting it, what is it?

The second option, the one I favour, would be to take Wittgenstein's disclaimers seriously: he is not trying to correct mathematics, nor trying to reject any part of it. Wittgenstein is actually quite insistent upon this point, repeating the message from different perspectives:

> *"Philosophy doesn't examine the calculi of mathematics, but only what mathematicians say about these calculi."* [13]

> *"I would say: 'I wouldn't dream of trying to drive anyone out of this [Cantor's] paradise'. I would try to do something quite different; I would try to show that this is not a paradise – so that you leave of your own accord."* [14]

> *"It is a strange mistake of some mathematicians to believe that something inside mathematics might be dropped because of a critique of the foundations. Some mathematicians have the right instinct: once we have calculated something it cannot drop out and disappear!"* [15]

> *"When set theory appeals to the human impossibility of a direct symbolization of the infinite it thereby introduces the crudest imaginable misinterpretation of its own calculus. (...) But of course that doesn't show the calculus to be something inherently incorrect (at most it shows it to be something uninteresting), and it's odd to believe that this part of mathematics is imperilled by any kind of philosophical (or mathematical) investigations. (With equal justification chess might be imperilled by the discovery that wars between two armies do not follow the same course as the battle on the chess board.)"* [16]

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13 BT, section 126 (p. 444e)

14 Lectures on the Foundations of Mathematics (p.103)

15 Wittgenstein and the Vienna Circle (p.149)

16 BT, section 137 (p. 495e)

What to do, then, with Wittgenstein's remarks about Euler's proof? What to do with his "constructivization"?

An interesting way to tackle the question would be to consider a most relevant problem in this respect: Is every mathematical proof liable to such a "correction", or to such "constructivization"? In their paper, Marion & Mancosu show that precisely this problem was being discussed by the time of the writing up of MS108, and that Heinrich Behmann, a student of Hilbert's who got acquainted with the work of Wittgenstein, actually believed to have found a general method of constructivization.

As it turns out, the question is a very delicate one. But we may assume, with the necessary caution needed in such cases, a negative answer: not all mathematical proofs admit a constructive version[17]. All the more so, not all mathematical proofs have built into them, as I tried to show with respect to Euler's proof, a hidden constructive structure.

It is precisely here that lies the advantage of seeing the *Standpunkt Wittgensteins*, not as an intromission in mathematics, but as something quite different. For if we interpret him as demanding a correction – say a constructivization of purported (albeit unwarranted) proofs –, such a correction may not be there to be recovered. This situation, however, would be utterly strange. It would be strange because, given any piece of mathematical work, it is almost a truism that *some* mathematical work has been done. Mathematicians, in this respect so much more practical than philosophers, have certainly agreed upon a lawful manipulation of signs. They understand each other reasonably well as regards such manipulations.

As always with Wittgenstein, it is all a matter of clarity. The calculus – any calculus – is not "imperilled by any kind of philosophical (or mathematical) investigations". We just need to correctly understand what it is that has been done.

This is the basic orientation, I believe, we should always assume while interpreting Wittgenstein's later work in the philosophy of mathematics. If we do that, then we will be able to focus on the two major problems that call, in this connection, for our attention. The task of the philosopher lies precisely in addressing them both. The first one is the relation between the language of mathematics (its symbolism) and quotidian language. The second one is

17 See, for example, [Paris & Kirby, 1982].

the relation, within mathematics, between its most basic part – the theory of numbers – and its more or less distant outgrowths[18].

To return to Euler's proof and Wittgenstein's treatment of it, I believe precisely these two problems are being dealt with.

On the one hand, the problem is a careless formulation of the final result, which fails to incorporate important determinations and, in so doing, treats the whole business of mathematics in a very unsatisfactory way: as if mathematics were a descriptive science, trying to reach some unreachable realm. What is now the source of this carelessness? As we may gather from the sections of the *Big Typescript* dealing with mathematics, it is an important one: a lack of attention to the particularities of mathematical language and the way it is used; and a related lack of sensitivity to the nature of mathematical language as purely normative (as *just* a calculus). This lack of sensitivity, in its turn, stems precisely from the belief that we can couch the results of mathematical work, without too much trouble, in quotidian language – a language mostly used in descriptive contexts. There we find, mixed with mathematics, the so-called mathematical prose, and the relation of this prose to the actual mathematical work is the real source of confusion Wittgenstein is trying to get rid of.

On the other hand, from this very same source of confusion comes a lack of attention to the inter-connections between the different parts of mathematics. For mathematics is made out – contrary to what we tend to believe, when we take its purpose to be the description of some mathematical realm – of a great variety of distinct calculi. Each calculus is, *per se* (that is: as long as it is a lawful manipulation of signs), plainly justified[19]. But we should pay the strictest attention – and we must always be capable of giving it a clear answer – to the following question: In what calculus is some result obtained and

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18 With respect to this, Wittgenstein offers a very interesting image: "Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow several metres long.)" BT, section 122 (p. 433e)

What Wittgenstein is complaining about is that, in present day mathematics, much calculus is done (none of them "wrong" or "imperilled by philosophy"), and few valuable mathematical products are obtained. Long shoots resulting in meagre potatoes, instead of (as good agriculture would have) short shoots with substantial potatoes to eat. But bad agriculture is still agriculture.

19 We may recall, in this connection, the interesting conversation between Wittgenstein, Waismann, and Schlick (the 30th of December 1930), where Wittgenstein observes that there is nothing wrong even with an "inconsistent" calculus (one permitting the derivation of "0≠0), as long as it is, qua calculus, a ruled manipulation of symbols (Wittgenstein und der Wiener Kreis, pp. 131-2 and 139). The point would certainly be the same for any "unconstructive" calculus (for example, one using some formal version of the excluded middle).

stated? The confidence in an all-embracing capacity of quotidian language to cut across the different calculi, and to be a common medium of expression for their results, is once again very tempting, and very misleading. The situation is particularly problematic, of course, with respect to the theory of natural numbers and its relation to other calculi, especially those calculi which contain formal versions of the excluded middle, *reductio ad absurdum*, double negation, and so on.

Euler's demonstration is, as we are now in a position to see, a veritable compendium of the aforementioned problems. No wonder Wittgenstein took so much trouble clarifying it. Accordingly, the "constructivization" is to be regarded, not so much as a correction of the proof (mathematical results do not need correction), but as Wittgenstein's minute effort to bring to the fore the working of such proofs; to see what they can really deliver, in a strict mathematical sense, without any trade with ordinary descriptive language; and to examine the connections between the different calculi of mathematics. It is to be regarded as Wittgenstein's minute effort to clarify the functioning of mathematics as pure calculus.

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