João Vergílio Gallerani Cuter*

Logic without Hierarchies[1]

Resumo

A lógica do Tractatus *não tem hierarquias. Todos os objetos (e consequentemente todos os nomes) estão no mesmo nível. Mesmo assim, a lógica possui categorias, com diferentes tipos de quantificadores associados a cada categoria. Muito embora o* Tractatus *não nos dê nenhum exemplo de proposição elementar, podemos imaginar que a análise foi levada até o final, e assim sublinhar algumas diferenças notáveis entre a lógica do* Tractatus *e aquela que encontramos nos manuais. Veremos que o* Tractatus *não tem nenhum problema em falar sobre um número infinito de objetos, muito embora não haja espaço para uma distinção entre diferentes tipos de infinitude. Mostramos como lidar com lidar com quantificações aninhadas no* Tractatus *sem introduzir elementos notacionais novos. Mostramos de que modo Wittgenstein acreditava ser possível expressar a equinumericidade (e também desigualdades) sem lançar mão de quantificadores de segunda ordem. Mostramos, finalmente, em que sentido o* Tractatus *pode ser inserido no contexto do projeto logicista.*

Palavras-chave: Wittgenstein . filosofia da linguagem . filosofia da lógica . Russell . teoria dos tipos . fundamentos da matemática.

Abstract

Tractarian logic has no hierarchies. All objects (and consequently all names) stand on the same level. Even so, it has logical categories, and different kinds of quantifiers attached to each category. Although the Tractatus itself does not give us any example of elementary proposition, we can imagine that analysis was completely carried out in order to stress some conspicuous differences between Tractarian logic

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 * Universidade de São Paulo (USP) / CNPq

and the logic we find in text-books. We will see that the Tractatus *has no problems to talk about an infinite number of objects, although there is no room in the book for different kinds of infinity. We show how to deal with nested quantifiers in the* Tractatus *without adding any notational element. We show how Wittgenstein believed he could express equinumerosity (and also inequalities) without appealing to second-order quantifiers. Finally we show in which sense the* Tractatus *can be inserted into the context of the logicist project.*

Tractarian logic is still more different from the logic we find in the ***Principia Mathematica*** than we are inclined to think[2]. The first and main difference is to be found at the level of names. The distinction between names and propositional functions became so familiar to us that it is difficult to conceive quantification theory against a different background. When we think about the opposition between names and propositional functions, we are probably thinking in Fregean terms. A propositional function is what is left of a proposition after the excision of one of its units of sense. It has an open place, or, as Frege used to say, it is "unsaturated". It asks for some kind of "complementation". In the usual cases, it asks for "names". Names and propositional functions have opposite and complementary natures. Names are saturated, and have no need of being "filled in". They have no "open places", no variables within.

In Russell, this conception is closely related to the idea of a "hierarchy" of expressions. The idea is roughly as follows. On the first level we have names

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2 Although the *Principia Mathematica* is the immediate target of the *Tractatus*, Wittgenstein's criticism can be extended to a quite general conception of "predication" that dates back to Aristotle. In the *Categories* (1a20-b9) we find the distinction between things which "are said" of other things (as man, for instance, is said of an individual man, like Socrates) and things which "are not said" of anything (individual entities in general). In our language, this distinction reappears in the form of terms that can be both subject and predicate (like "man"), and terms that can only appear in the place of a grammatical subject (like "Socrates"). There is an obvious analogy between this Aristotelian dichotomy and the Fregean distinction between "saturated" and "unsaturated" entities and expressions. Russell's theory of types is built by its turn on the basic opposition between individuals, on the one hand, and all kinds of propositional functions, on the other. All these dichotomies involve some kind of "hierarchy", and so they are bound to fall under the scope of the Tractarian criticism sketched below.

of individuals. They are "saturated", since they do not have "open places" — they are not to be "completed" by any expression. On the second level we have "unsaturation" of a certain kind — we have propositional functions taking names of individuals as arguments. On the third level we have propositional functions whose arguments are functions of the second level, and so on. This hierarchy constitutes the core of Russell's theory of types — the axis around which its many "ramifications" will be defined. The theory of types is Russell's logical syntax of language.[3] The most fundamental rule of this syntax says that a propositional function cannot take functions of its own level as arguments. In the case of predicative functions, they must look for completion at the level immediately below itself in the hierarchy.

Wittgenstein's logic has no hierarchies. At the basis of language, we have the totality of logically proper names. Each of these names must be given with definite syntactical rules attached to it, and these rules must mirror the combinatorial possibilities of the named object. If an object can combine with another, their names can also combine; if they can't, their names can't either. That's all. No hierarchical positions have any place here. All names are so to speak on the same level. In Wittgenstein's logic it is perfectly possible to imagine that f_1a, f_2a , f_3a, f_4a, and $f_1f_2f_3f_4$ are all meaningful propositions, without the risk of being prey to paradoxes. (Indeed these are propositions of the hypothetical language we introduce below.)

Any possible (i.e., syntactically permitted, i.e., ontologically grounded) combination of names is an elementary proposition. An elementary proposition is the description of a "state of affairs", that's to say, of an immediate concatenation of objects. The totality of elementary propositions (in ontological terms, of "possible states of affairs") is the "logical space" — the total space of factual possibilities. As Wittgenstein does not give us a single example of elementary proposition, we will have to imagine that logical analysis has been effected (along strictly Tractarian lines, of course), and that the "logical space" was finally reached.

So let us imagine it.

Our hypothetical logical space will be defined by the possible concatenations of objects belonging to three logical categories. The first one has only

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3 This is only true in so far as "formation rules" are concerned: the theory gives the formation rules for the elementary proposition, and also includes the rules regulating the use of connectives and quanfiers (cf. *Principia Mathematica*, *9.131). The so-called "transformation rules" naturally lie outside the scope of the theory.

two objects – let us call them[4] *a* and *b*. The second has five, named *A, B, C, D, E*. The third category has an infinite number of objects, and we associate an infinite number of names to it: $f_1, f_2, f_3,...$ These categories are associated to possibilities of concatenation giving rise to states of affairs. We can easily describe these possibilities if we define certain variables to range over each of the categories, establishing from the start that different kinds of variables will range over different kinds of objects, and different variables (of a kind) over different objects (of the same kind). The variables α and β will range over objects of the first, $\Gamma, \Delta, \Lambda, \Pi$ and Σ over objects of the second, and $\varphi_1, \varphi_2, \varphi_3$, etc. over objects of the third category.[5] Now any possible elementary proposition will have one of the following logical forms: $\varphi_1 \alpha$, $\varphi_1 \varphi_2 \Gamma$, or $\varphi_1 \varphi_2 \varphi_3 \varphi_4$. Accordingly $f_1 b$, $f_1 f_2 A$ and $f_1 f_2 f_3 f_4$ will be meaningful propositions of our language, while $f_1 A$, $f_1 f_2 b$, $f_1 f_1 B$, $f_1 f_2 f_3$ and $f_1 f_1 f_2 f_3$ are to be taken as meaningless sequences of signs.[6] Moreover, it must be understood that $f_1 f_2 f_3$ is not a possible value of $f_1 f_2 \Gamma$. The values of this function are $f_1 f_2 A$, $f_1 f_2 B$, $f_1 f_2 C$, $f_1 f_2 D$ and $f_1 f_2 E$.

An elementary propositional function is obtained from an elementary proposition through substitution of variables for names. Any such function is associated with the totality of its values, and in their relation to that function these values are said to be "formally determined". If these totalities of values are taken as totalities of signs (as opposed to symbols), they can be described. As they are formally determined, we have (so to speak) a "recipe" to build

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4 In all contexts involving the mention of a symbol I will make an *autonimous use* of the symbol, i.e. in these contexts I will not use quotation marks in order to make reference to the symbol itself.

5 It should be observed that if an ontological category has only a finite number of objects, the corresponding syntactical category must have the *same* (finite) number of names, and language will incorporate exactly one variable for each name. This is an immediate consequence of the convention establishing that different variables range over different objects. $f_1 a$ & $f_2 a$, for instance, is *not* a possible value of the propositional function $f_1 a$ & $f_2 \beta$. For exactly the same reason, we could not introduce a third variable in the first category: $f_1 \alpha$ & $f_2 \beta$ & $f_2 \gamma$ would have no value *at all*. That is not a logical form of our hypothetical logical space.

6 In the first two cases, because a name of the wrong category is being used in the last position. In the fourth case because the "arity" is wrong — "$f_1 f_2 f_3$" is not a proposition of our hypothetical language exactly by the same reason that "John is to the right of" is not a proposition of everyday language. In the second and fifth cases because the same name appears in different positions, something which is excluded by the formation rules we have laid down. I will therefore assume that no "repetition of names" is admissible inside an elementary proposition of our hypothetical language. Although I do think that this assumption would have to hold in general for the *Tractatus*, I won't give arguments here to make this point. I am just taking "non--repetition" as a special feature of the hypothetical language under consideration.

these signs. Thus, for instance, f_1a and f_1b is a formally determined totality of propositions, since these are all the values of f_1a. By the same token f_1a, f_1b, f_2a, f_2b, f_3a, f_3b, f_4a,…etc. is a formally determined totality as well: these are all the values of the propositional function φ_1a. The only difference is that in this case we have an infinite totality — a circumstance indicated by the use of the three dots followed by the expression "etc.". As this infinite totality is presupposed in the use of the variable itself, it must be viewed as *formally* determined. It corresponds to a logical form of our language.

The simplest way of determining a totality of propositions is to enumerate them. Unfortunately it only works when we are dealing with finite totalities. We may use the Greek letter ξ as an ancillary variable ranging over the members of any formally determined totality. A bar over a variable will indicate that the totality of its values is under consideration. The symbol

$$\overline{\xi} = f_1b, f_1f_2D$$

will mean that f_1b and f_1f_2D are all the values of ξ to be considered in a given context. Accordingly, if we write

$$\overline{\xi} = \overline{\varphi_1a}$$

we mean that the propositions in the scope of the propositional variable ξ are all the propositions which are values of the function φ_1a. Finally, we may combine both methods and write, for instance

$$\overline{\xi} = \overline{\varphi_1\varphi_2\Gamma}, \ \overline{\varphi_1\varphi_2\varphi_3\varphi_4}$$

In this case, both f_2f_3B and $f_2f_1f_5f_3$ will be among the infinite number of propositions falling within the scope of the variable ξ.

By means of direct enumeration and simultaneous denial we can define any truth-operation over a finite number of operands. Let

$$\overline{\xi} = p_1, ..., p_n$$

and

$$\text{N}' \left[\overline{\xi}\right]$$

be the simultaneous negation of all propositions within the scope of ξ. It is clear that $N'[N'[\bar{\xi}]]$ will be equivalent to a disjunction of all the propositions enumerated. On the other hand, if we make

$$\bar{\xi} = N'[p_1] \ ..., N'[p_n]$$

then $N'[\bar{\xi}]$ will be equivalent to a conjunction of $p_1, ..., p_n$.

Quantification makes its entrance as soon as we adopt the second method of formal selection, that is to say when we use the propositional variable ξ to range over all the values of a propositional function. Let

$$\bar{\xi} = \overline{\varphi_1 a}$$

In this case, $N'[N'[\bar{\xi}]]$ will be the proposition $(\exists \varphi_1)\varphi_1 a$ - a disjunction of the infinite (but formally determinate) totality of propositions $f_1 a, f_2 a..., f_n a$, *etc.* The conjunction of these propositions can be analogously achieved, giving rise to universal quantification.

Quantifiers can be defined, but not eliminated. They were defined for the most basic situations alone. When iterated, they must be actually present. No contortion will give us $(\exists \varphi_1)(\forall \varphi_1)\varphi_1 \varphi_2 A$ without passing through the infinite totality

$$(\forall \varphi_2)f_1\varphi_2 A, \ \ (\forall \varphi_2)f_2\varphi_2 A, \ ..., \ \ (\forall \varphi_2)f_n\varphi_2 A$$

This totality is formally determinate only because it is the totality of values of a propositional function, viz.

$$(\forall \varphi_2)\varphi_1 \varphi_2 A$$

The original proposition can be written using the definition of the existential quantifier in terms of formal selection and simultaneous denial:

$$N'[N'[\overline{(\forall \varphi_2)\varphi_1 \varphi_2 A}]]$$

Eliminating the universal quantifier, we get

$$N'[N'[N'[\overline{N'[\varphi_1 \varphi_2 A]}]]]$$

Which is equivalent to $N'[N'[\varphi_1\varphi_2A]]$, *i.e.* $(\forall\varphi_1)(\forall\varphi_2)\varphi_1\varphi_2A$. And it is obvious that $(\exists\varphi_2)(\forall\varphi_1)\varphi_1\varphi_2A$ *and* $(\forall\varphi_1)(\forall\varphi_2)\varphi_1\varphi_2A$ say quite different things. We could circumvent the problem marking the variable to be considered in each step, but this is exactly the task variables are supposed to do. Marked variables will only give us quantifiers under an awkward disguise. I'd rather use them openly.

Recognizing quantifiers as essential logical tools is one thing; admitting the existence of propositions that could not be built from elementary ones with the help of simultaneous denial and the mechanisms of formal selection is quite another. The whole problem is that formal selection must be made step by step. Let us take the elementary proposition f_1f_2A as our point of departure. Selecting this proposition alone (i.e., by direct enumeration), we arrive at a formally determinate totality – $[f_1f_2A]$ – to which we can apply simultaneous denial: $N'[\overline{f_1f_2A}]$. More briefly, $N'[f_1f_2A]$. Substituting a variable for the name f_2, we get a propositional function and a new totality associated to it. Applying simultaneous denial to this totality, we arrive at $N'[\overline{N'[f_1\varphi_2A]}]$. Now we can freeze this step using a quantifier

$$(\forall\varphi_2)f_1\varphi_2A$$

and proceed without ambiguities.

The most original logical device to be found in the *Tractatus* is formal selection of a "series of forms". A totality of propositions is given by a "series of forms" when it can be inductively characterized. We can express the induction rule by means of an "expression in brackets" of the form $[p, x, O'x]$, where p is the first proposition, x is any proposition in the series, and O is the formal procedure giving the "successor" of x. Negation provides the simplest example. The series

$$p$$
$$N'[p]$$
$$N'[N'[p]]$$
$$etc.$$

can be written as $[p, x, N'x]$. We may call this a "dull series", since it unfolds itself by a monotonous alternation of two propositions. Even so, it is a series

of forms. We can select the whole series writing a bar over the expression in brackets. In this case,

$$N'[\overline{p, x, N'x}]$$

is a contradiction, while

$$N'[N'[\overline{p, x, N'x}]$$

is a tautology.

We can also write the "dull series" given above using exponents to count the number of times the operation N was applied to the basis:

$$N^{0'}[p]$$
$$N^{0+1'}[p]$$
$$N^{0+1+1'}[p]$$
$$etc.$$

It is obvious that we are not using numbers to count apples or trees. We are just counting how many times an operation was applied to a basis, or (more exactly) expressing a proposition as a result of applying a recursive operation a certain number of times. It is important to notice that no quantifiers are involved in this kind of counting. As Wittgenstein says, numbers are used in language as "exponents of operations" – abbreviations made possible by the recursive nature of an operation like N.

Arithmetical operations can be defined using the logical operation N alone. We first define the symbol $N^{v'}x$ (where the variable x stands for whatever proposition we like):

$$N^{0'}x = x$$
$$N^{v+1'}x = N'N^{v'}x$$

Then we define addition:

$$N^{v+\mu'}x = N^{v'}N^{\mu'}x$$

By hypothesis, x is a proposition, and so both $N^{\mu\prime}x$ and $N^{\nu\prime}N^{\mu\prime}x$ must be propositions as well. Using the decimal system of abbreviation, we have, e.g.,

$$N^{2+0\prime}x = N^{2\prime}N^{0\prime}x = N^{2\prime}x = N'N'x$$

Now we can use the symbol $(N^{\nu})^{\mu\prime}x$ to express the result of μ applications of the (complex) operation N^{ν} over the basis x. More exactly,

$$(N^{\nu})^{0\prime}x = x$$
$$(N^{\nu})^{\mu\prime}x = N^{\nu\prime}(N^{\nu})^{\mu\prime}x$$

And we can define

$$N^{\nu\times\mu\prime}x = (N^{\nu})^{\mu\prime}x$$

Here are two examples of how these definitions can be applied to "prove" arithmetic properties:

a) $N^{3\times0\prime}x = (N^3)^{0}{}'x = x = N^{0\prime}N^{0\prime}N^{0\prime}x = (N^0)^{3}{}'x = N^{3\times0\prime}x$

b) $N^{3\times2\prime}x = N^{3\prime}N^{3}N^{0\prime}x = N^{3\prime}N^{3\prime}x$
 $= N'N'N'N^{0\prime}N'N'N'N^{0\prime}x$
 $= N'N'N'N^{0\prime}N'N'N'x$
 $= N'N'N'N^{0\prime}y \ (where \ y = N'N'N'x)$
 $= N'N'N'y$
 $= N'N'N'N'N'N'N^{0\prime}x$
 $= N^{6\prime}x$

Some general ("algebraic") laws such as

$$N^{\nu+(\mu+\lambda)\prime}x = N^{(\nu+\mu)+\lambda\prime}x$$

can be established directly: $N^{\nu+(\mu+\lambda)\prime}x = N^{\nu\prime}N^{\mu+\lambda\prime}x = N^{\nu\prime}N^{\mu\prime}N^{\lambda\prime}x = N^{\nu\prime}N^{\mu\prime}y$ (where y is the proposition $N^{\lambda\prime}x$) $= N^{\nu+\mu\prime}y = N^{\nu+\mu}N^{\lambda\prime}x = N^{(\nu+\mu)+\lambda\prime}x$. Other laws can only be proved by induction, e.g. $N^{\nu+\mu\prime}x = N^{\mu+\nu\prime}x$.

 As applied to a single proposition, N is the usual denial operation. Using the logical properties of this operation (as shown in the corresponding truth-table), we could prove such general laws as $(N^2)'x = x$. This is simply another

way of saying that couples of contiguous negations cancel each other. Now let us consider another operation – an even "duller" one. We may call it T. If x is a proposition, then $T'x$ will be true no matter the truth-value of x. Now it is obvious that $(T^2)^{\nu\prime}x = x$ will not hold, while $T^{\nu+\mu\prime}x = T^{\mu+\nu\prime}x$ will be in perfect logical order, provided we repeat for T the definitions of the exponents and operations on exponents we built for N. These definitions are not essentially linked to any particular operation. They just give expression to the general idea of "recursion". For let O be any recursive operation you want to consider – be it N, T, or any other whatsoever. When we define the symbol $O^{\nu\prime}x$ we give expression to the circumstance that recursive operations can be indefinitely applied to their own results. The same can be said of a symbol like $O^{5\prime}O^{7\prime}x$. Have we applied the operation O seven times? Now we may take this partial result as a basis to begin a new series of five successive applications of the same operation. "Five plus seven" is just a short way of saying that. Numbers are exponents of recursive operations, and arithmetic is the general "theory" (better to say "technology") of recursion.

Is it? As we said, numbers are used to count apples and sheep, not successive performances of operations like N. Dull operations like N and T do not even correspond to any use of numbers in ordinary language. They are "dull" exactly because they give rise to dull "series of forms". Let us take "it is raining" as our basis. In English, the series $p, N'p, N'N'p, etc.$ would be written more or less like this: "It is raining", "It is not true that it is raining", "It is not true that it is not true that it is raining", etc. Dullness apart, the fact is that we cannot find any number at work here. In our everyday language, we never use the word "zero" to say that it is raining, nor the word "one" to say the contrary.

We use numbers to count objects – if not apples and books, at least our "logically simple" objects $f_1, f_2, ..., f_n, ..., A, ..., E, a$ and b. Wittgenstein's strategy is to show that counting objects is just a special case of counting successive performances of logical operations. Given the Tractarian convention governing the use of variables (different variables, different arguments), the propositions of the series

$$\sim(\exists\varphi_1)\varphi_1 a$$

$$[\sim(\exists\varphi_1)(\exists\varphi_2)\varphi_1 a . \varphi_2 a] . [(\exists\varphi_1)\varphi_1 a]$$

$$[\sim(\exists\varphi_1)(\exists\varphi_2)(\exists\varphi_3)\varphi_1 a . \varphi_2 a . \varphi_3 a] . [(\exists\varphi_1)(\exists\varphi_2)\varphi_1 a . \varphi_2 a]$$

etc.

say that "there is no object φ_1 such that $\varphi_1 a$", "there is exactly one object φ_1 such that $\varphi_1 a$", "there are exactly two objects φ_1 and φ_2 such that $\varphi_1 a$ and $\varphi_2 a$", etc. It will be convenient to use a more economical notation for series like this:

$$(E0\ \varphi_1)\ \varphi_1 a$$
$$(E1\ \varphi_1)\ \varphi_1 a$$
$$(E2\ \varphi_1)\ \varphi_1 a$$

etc.

The series can be recursively defined: it unfolds according to a definite transformation rule. Let this rule be called O_1. The series of forms can now be written in the "square brackets" notation:

$$[(E0\varphi_1)\varphi_1 a, x, O_1'x]$$

Accordingly,

$$[(E5\varphi_1)\varphi_1 a, x, O_1'x]$$

will begin affirming the existence of five such objects, and then proceed to six, seven, eight, and so on. In both cases, the transformation rule is exactly the same - O_1.

    Consider now the series

$$(E0\varphi_1)\varphi_1 a\ .\ (E0\varphi_1)\varphi_1 b$$

$$(E1\varphi_1)\varphi_1 a\ .\ (E1\varphi_1)\varphi_1 b$$

$$(E2\varphi_1)\varphi_1 a\ .\ (E2\varphi_1)\varphi_1 b$$

etc.

and call O_2 the transformation rule involved in it. Writing p for $(E0\varphi_1)\varphi_1 a\ .\ (E0\varphi_1)\varphi_1 b$, the expression in brackets will be

$$[p, x, O_2'x]$$

and the proposition

$$N'[N'[\overline{p,\ x,\ O_2'x}]]$$

will say that one proposition of the series is true. In other words, the proposition as a whole says that the number φ_1's of 's such that $\varphi_1 a$ is identical to the number of φ_1's such that $\varphi_1 b$.

This is perhaps the most interesting feature of Wittgenstein's logic. It does not have any place for "higher order" quantifiers. All quantifiers are, so to speak, on the same level: they all range over objects of a certain kind, there being no "hierarchy" to stratify the kinds. But language does not loose the power of expressing logical relations that were expressed in the logic of Frege and Russell through the use of second-order quantifiers. We may say that "the number of… is identical to the number of…" without having to affirm the existence of a "relation" obeying to such and such conditions. We just have to (i) specify a series of forms by means of its first member and its transformation rule, (ii) select the totality of members of the series, and (iii) apply a truth-operation over this totality.

Let me give two other examples, showing how far this method can lead us. The expressions "there is at least one…", "there are at least two…", there are at least three…", etc. can be used to build a series of forms, e.g.

$$(\exists\varphi_1)\varphi_1 b$$

$$(\exists\varphi_1)(\exists\varphi_2)\varphi_1 b \,.\, \varphi_2 b$$

$$(\exists\varphi_1)(\exists\varphi_2)\,(\exists\varphi_3)\varphi_1 b \,.\, \varphi_2 b \,.\, \varphi_3 b$$

etc.

More briefly,

$$(\exists 1\varphi_1)\varphi_1 b$$

$$(\exists 2\varphi_1)\varphi_1 b$$

$$(\exists 3\varphi_1)\varphi_1 b$$

etc.

Now we form a compound series:

$$(\exists 1\varphi_1)\varphi_1 b \, . \, (E0\varphi_1)\varphi_1 a$$

$$(\exists 2\varphi_1)\varphi_1 b \, . \, (E1\varphi_1)\varphi_1 a$$

$$(\exists 3\varphi_1)\varphi_1 b \, . \, (E2\varphi_1)\varphi_1 a$$

etc.

Let O_3 be the transformation rule governing the development of this series, and let q be the proposition $(\exists 1\varphi_1)\varphi_1 b \, . \, (E0\varphi_1)\varphi_1 a$. The proposition $N'[N'[\overline{q, x, O_3'x}]]$ says that the number of φ_1's concatenated with b is greater than the number of φ_1's concatenated with a. The proposition is not asserting that there is a finite number of φ_1's concatenated with b, but it implies that there is a finite number of φ_1's concatenated with a. As a matter of fact, when we affirm that at least one of the propositions of the series

$$(\exists 0\varphi_1)\varphi_1 a$$

$$(\exists 1\varphi_1)\varphi_1 a$$

$$(\exists 2\varphi_1)\varphi_1 a$$

etc.

is true, we are affirming that there is a finite number of φ_1's concatenated with a. Simultaneous denial of all propositions in this series says that there in an "infinite number" of φ_1's concatenated with a.

So we may talk about an "infinite number of objects". In spite of this, "infinite" is not a kind of number, and there is no place for a hierarchy of different kinds of infinity. Numbers always appear as exponents in the truth-functional building of complex propositions, and the number of steps in any such building is always finite. We know, for instance, that $O_3^2\,'q$ is (according to the conventions given above) the proposition

$$(\exists 3\varphi_1)\varphi_1 b \, . \, (\exists 2\varphi_1)\varphi_1 a$$

while $O_3^{*\prime}q$, understood as a shorthand for the (supposed) proposition that (supposedly) comes after all propositions in the series [q, x, $O_3'x$], has no meaning at all.

A special (and important) feature of our hypothetical language is that quantification is limited by the possible values of the quantified variable. I can write $(\exists\alpha)f_1\alpha$ meaning $f_1a \vee f_1b$, but $(\exists 3\alpha)f_1\alpha$ has no meaning at all. I don't have in my language three different variables ranging over the objects a and b. Adding a new variable γ will not do, since a "function" like $f_1\alpha$. $f_1\beta$. $f_1\gamma$ will have no values. I simply run out of names. If you say "Just add new names to the language!", Wittgenstein's answer would be - "Meaning what?".

Better not to use quantifiers when we have a limited stock of names. Usual truth-functions can do the same work at no risk. The propositions $\sim(\exists\alpha)f_1\alpha$, $(\exists 1\alpha)f_1\alpha$ and $(\exists 2\alpha)f_1\alpha$ can be written as $\sim f_1a.\sim f_1b, f_1a \vee f_1b$ and $f_1a f_1b$. Analogous cases will be handled analogously.

It obvious that arithmetic must not wait for the final analysis of language in order to know if it has logical foundations or not. Even if there were just two objects a and b, and a solitary possibility in the logical space, even so we should be able to build series like

$$ab$$
$$\sim \sim \sim ab$$
$$\sim \sim \sim \sim \sim \sim ab$$

etc.

Let O_4 be the operation governing the series. The proposition O_4^{12} will keep being identical to the proposition $O_4^{5+7\prime}ab$, and "5 + 7 = 12" will keep being a correct logical rule. Arithmetic is a "method of logic". It abbreviates that part of logic involved in recursive logical operations like N, T, O_1, O_2, O_3, or any other operation defined in terms of truth-functions and formal methods of selection. This was Wittgenstein's early version of logicism.