On Conservative and Expansive Extensions

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Abstract

Conservativeness is an important property of extensions. Its model-theoretic counterpart, expansiveness, provides a useful sufficient condition for conservativeness, but they are not equivalent, even under very severe restrictions. This paper examines the causes of this phenomenon, emphasizing extensions by constants. Some simpler characterizations for conservativeness are provided and finite conservative extensions are shown to be equivalent to Skolem extensions.

Keywords :

Conservative extensions, model expansion, Skolem extensions, extensions by constants, Mathematical Logic, formal specifications.

Resumo

Uma propriedade importante de uma extensão é ser conservativa. Sua contrapartida em termos de modelos, expansividade, fornece uma condição suficiente bastante útil para uma extensão ser conservativa, mas estes dois conceitos não são equivalentes, nem mesmo sob restrições bastante severas. Este trabalho examina as causas desse fenômeno, enfatizando extensões por constantes. Apresentam-se algumas caracterizações simplificadas de extensões

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conservativas e mostra-se que as extensões conservativas finitas são equivalentes às extensões de Skolem.

Palavras chave :
Extensões conservativas, expansão de modelos, extensões de Skolem, extensões por constantes, Lógica Matemática, especificações formais.

1. Introduction

This paper examines conservative extensions and those that satisfy the usual model-expansion criterion for conservativeness, emphasizing extensions by constants, even though some of the results can be extended to the case of addition of function symbols, as well. The causes of failure of this criterion are illustrated and examined, and then conservative extensions by constants are characterized. Finally, the finite conservative extensions by constants are shown to be exactly the Skolem extensions, which do satisfy the above model-theoretic criterion.

The concept of conservative extension is very important in mathematical logic, many of its uses, for instance in proving relative consistency, being due to the fact that they provide a way of preserving consistency (see, e.g. [Enderton '72; Shoenfield '67; Smirnov '86] . They can also be regarded as encompassing several useful generalizations of the familiar extensions by definitions [Veloso + Veloso '90] .

Conservative extensions are also often used in the context of formal specifications [Byers + Pitt '90, p. 196] . They are employed, for instance in [Ehrig + Mahr '85; Turski + Maibaum '87; Maibaum + Veloso + Sadler '84; Veloso '87], for dealing with program development by stepwise refinement and abstract data types. They are also useful in formalizing some important intuitive ideas [Polya '57] concerning problems, solutions, analogy and problem-solving methods [Veloso + Veloso '81; Veloso '84; Veloso '88].

There are two natural ways of formalizing this concept, namely in terms of either the theories themselves or their models. It has been known that, in first-order logic, the semantical version is strictly stronger than the other one. Let us call « non-smooth » those extensions that are conservative according to the first version but not according to the second one. In this paper we examine more closely the causes of this phenomenon, showing some typical ways to obtain non-smooth extensions and characterizing the conservative extensions by constants.

The structure of this paper is as follows. The next section reviews some basic concepts, notations and results. Section 3 examines the causes of non-expansiveness by presenting and illustrating some typical ways of obtaining non-smooth extensions. Then, section 4 presents some model-theoretic properties of conservative extensions and uses them to investigate the behavior of finite models. In section 5 we provide a simple characterization for the conservative extensions by constants. Then, section 6 characterizes the finite conservative extensions by constants as those obtained by the addition of Skolem constants, which are expansive. Finally, section 7 presents some concluding remarks and comments.

2. Preliminaries

Let us first review some basic concepts and notations to be employed throughout the paper (for more details, see, for instance, [Shoenfield '67; Enderton '72; van Dalen '89] .)

By a *language* L we mean a first-order (possibly many-sorted [Enderton '72, p. 277]) language consisting of logical and extra-logical symbols [Shoenfield '67, p. 14; van Dalen '89, p. 61] . We generally assume L to include the logical symbol $\simeq$ for identity. As usual, a *structure* A for L consists of an assignment of a realization A [s] to every extra-logical symbol s, respecting the syntactical declarations [Shoenfield '67, p. 18; Enderton '72, p. 79] . We say that language L' is a *sub-language* of language L'' (which we denote by L' $\subseteq$ L'') iff L'' can be obtained from L' by the addition of some new extra-logical symbols (together with appropriate syntactical declarations). If L' $\subseteq$ L'' and the structure A' for L' is the restriction to L' of the structure A'' for L'', then we call A' the *reduct* to L' of A'' and use A'' $\restriction$ L' to denote A'; in this case we also call A'' an *expansion* (to L'') of A' [Shoenfield '67, p. 43; van Dalen '89, p. 116] .

By a *theory* (*presentation*) T we shall mean a language L together with a set S of *axioms,* which are sentences of L; we often say that T is *over* L. A structure A is said to be a *model* of T (denoted by A $\models$ T) iff A is a structure for L that satisfies all the axioms of T (denoted by A $\models$ S). We use the notation T $\models$ F (or S $\models$ F) to state that F is a logical *consequence* of T (or of S) and Cn(T) for the set of *consequences* of T. By the *theory* of a structure A for L we mean the set Th(A) of all sentences of L holding in A.

We say that T'' (over L'' with set of axioms S'') is an *extension* of T' (over L' with set of axioms S'), which we denote by T' $\subseteq$ T'', iff L' $\subseteq$ L'' and Cn S' $\subseteq$ Cn S''. Also, T' and T'' are called *equivalent* iff they are extensions of each other; notice that, in this case, they have the same language. We call an extension T' $\subseteq$ T'' *conservative* iff for every sentence F of L', whenever T'' $\models$ F then T' $\models$ F as well [Shoenfield '67, p. 41, 42] .

One of the motivations for conservative extensions is the fact that they preserve consistency. Indeed, the following remark should be clear [Shoenfield '67, p. 42] .

Remark. *Conservativeness and consistency*

. If the extension T′ ⊆ T″ is conservative, then T′ is consistent iff T″ is consistent.

One can characterize conservative extensions by means of the concept of restriction of a theory. Given a theory T″ over L″ and a sub-language L′ ⊆ L″, by the *restriction* of T″ to L′ we mean the set of all sentences of L′ that are in Cn(T″) [Shoenfield '67, p. 95 (exercise 9)]. Then, it is easy to see that the extension T′ ⊆ T″ is conservative iff T′ is equivalent to the restriction of T″ to L′. In fact, the restriction T′ of T″ to L′ is, up to equivalence, the smallest theory over L′ such that the extension T′ ⊆ T″ is conservative.

Unfortunately the above characterization of conservative extensions in terms of restrictions is not very useful, since it amounts to a simple restatement of the definition. A more convenient sufficient condition for conservativeness is provided by means of reducts, which may be regarded as model-theoretic counterparts of restrictions. We then have a simple model-expansion criterion [Shoenfield '67, p. 65 (exercise 3.b)]. In order to formulate it more clearly, we introduce the following concept. We shall call an extension T′ ⊆ T″ *expansive* iff every model **A′** of T′ can be expanded to a model **A″** of T″.

Proposition. *Model-theoretic criterion for conservativeness*

If the extension T′ ⊆ T″ is expansive then it is conservative.

Some simple properties of these concepts are given in the next immediate result.

Lemma. *Properties of conservative / expansive extensions*

Consider extensions T ⊆ T′ ⊆ T″.

(a) If extension T ⊆ T″ is conservative (resp. expansive), then so is T ⊆ T′.
(b) If T ⊆ T′ and T′ ⊆ T″ are conservative (resp. expansive), then so is T ⊆ T″.

Simple and well-known examples of conservative extensions are the extensions by definitions. An extension T′ ⊆ T″ by definitions actually satisfies a stronger version of the above model-theoretic criterion, in that each model of T′ has a unique expansion to a model of T″ [Shoenfield '67, p. 60, 61] .

We shall often have occasion to write a formula F in the form $F(x_1, ..., x_m, y_1, ..., y_n, z_1, ..., z_p)$ when we wish to emphasize the fact that the displayed variables are among those with free occurrences in F. In this case, given constants $c_1, ..., c_n$, $F(x_1, ..., x_m, c_1, ..., c_n, z_1, ..., z_p)$ is the result of the simultaneous replacement of every free occurrence of y_j in F by c_j, for j = 1, ..., n.

3. Non-expansiveness and its causes

In this section we wish to examine some aspects of non-expansiveness. First, let us call an extension *non-smooth* iff it is conservative but not expansive. Thus, the main aims of this section are :

1. to establish the existence of non-smooth extensions;
2. to exhibit some typical examples of this phenomenon;
3. to show some canonical ways of obtaining such extensions;
4. to examine the causes of this phenomenon.

The main problems faced in obtaining a non-smooth extension T″ of T′ turn out to be guaranteeing (i) conservativeness and (ii) non-expansiveness, for one cannot rely on the simple model-expansion criterion of the proposition in section 2. The solutions we propose are, for (i), make T′ complete, and, for (ii), make T″ force its models to be « large enough ». By a *complete* theory we mean a maximally consistent one, in that, for any sentence F of its language, either F or its negation is a consequence of the theory [Enderton '72, p. 145] .

The following simple lemma characterizes the conservative extensions of a complete theory. We shall have occasion to use it quite often in the sequel, in view of our suggestion above to ensure conservativeness.

Basic lemma. *Conservative extensions of complete theories*

If T′ is complete, then an extension T″ of T′ is conservative iff it is consistent.

Proof.

($\Rightarrow$) Well known (see the remark in section 2).

($\Leftarrow$) Consider a sentence F of L′. If $F \notin Cn(T')$, since T′ is complete, $\neg F \in Cn(T')$, whence $\neg F \in Cn(T'')$, so $F \notin Cn(T'')$, since T″ is consistent. QED

The structure of this section follows a pattern that should be made clear. We present five typical examples. Each one of them can be generalized to a canonical way of obtaining non-smooth extensions. But, against each example some objections can be raised. These objections are faced in the next example, until we reach the end of the section, where we show a general way of obtaining non-smooth extensions provided the original theory has a somewhat « infinite » nature. We shall generally consider the language L′ of the original theory T′ to be countable. This is for the sake of simplicity only; the examples and arguments are easily carried over to other infinite cardinalities.

3.1. L″ much bigger than L′

We shall first establish the existence of non-smooth extensions, without worrying about their cardinalities.

Example 1.

Consider a countably infinite structure **Q** in an appropriate language L′ and let T′ = Th(**Q**) (For instance, take **Q** as the rational numbers).

The set of axioms to be added is as follows. Consider an uncountable set C of new constants and set L″ = L′ ∪ C. Now let Expld(L′) be the set of sentences of the form $\neg\, c \doteq d$ for all distinct constants c and d in C, and take T″ = T′ ∪ Expld(L′).

A similar example was independently presented by [Byers + Pitt '90] . Actually, it turns out to be an instance of a general phenomenon, to be presented in the next proposition, the proof of which will establish that T″ = T′ ∪ Expld(L′) is a non-smooth extension of T′.

Proposition 1.

Any complete theory T′ with infinite models has some non-smooth extension.

Proof.

Pick an uncountable set C of new constants and form L″ = L′ ∪ C. Now let Expld(L′) consist of the sentences $\neg\, c \doteq d$ for all distinct constants c and d of C, and set T″ = T′ ∪ Expld(L′).

Notice that any model of T″ must be uncountable.

But, in view of Löwenheim-Skolem theorem, T′ has some countably infinite model A. Such a model A of T′ cannot be expanded to a model of T″.

On the other hand, T″ is consistent, as can easily be seen by compactness (any finite subset of Expld(L′) can be satisfied in an appropriate expansion of A).

Therefore, by our basic lemma, we have that T″ is a non-smooth extension of T′.

QED

In view of the above, we have established

Conclusion 1.

There exist non-smooth extensions.

But, the example presented may be deemed somewhat contrived.

Objection 1.

The cardinality of L″ is much bigger than that of L′.

3.2. L″ and L′ with the same cardinality

We now face objection 1 raised above by examining non-smooth extensions where both languages have the same cardinality.

Example 2.

Let us consider the structure **N** of the natural numbers with zero and successor in an appropriate language L′ with *0* and *S*. We take T′ = Th(**N**). The set of axioms to be added is defined as follows. For each natural number n ∈ N, let *n* be the variable-free term *S* ... *S 0* (with n occurrences of *S*). We pick a new constant *c* and add it to L′ to form L″. We now set Unnam(L′) = {¬ *c*≃*n* / n ∈ N} and T″ = T′ ∪ Unnam(L′).

This example is again an instance of a more general phenomenon, to be dealt with in the next proposition, whose proof will establish that T″ = T′ ∪ Unnam(L′) is a non-smooth extension of T′. For this purpose, it is convenient to introduce some terminology and notation. We denote by Nam(L′) the set of all variable-free terms of language L′. We shall call a structure for L′ *named* iff every element of its domain is the value of a variable-free term [Veloso '79; Veloso + Veloso '89]. We can now present a generalization of our example 2, where we shall see that this example relies simply on the fact that the standard model of the natural numbers is an infinite named structure.

Proposition 2.

Any complete theory T′ with an infinite named model has some non-smooth extension.

Proof.

Pick a new constant *c* and add it to L′ to form L″. Now consider Unnam(L′) = {¬ *c* ≃t / t ∈ Nam(L′)}, and, finally, set T″ = T′ ∪ Unnam(L′). Consider an infinite named model A of T′.

Notice that model A of T′ cannot be expanded to a model of T″, because A is named.

But, T″ is consistent, as can easily be seen by compactness (any finite subset of Unnam(L′) can be satisfied in some expansion of A).

Therefore, by the basic lemma, we can conclude that T″ is a non-smooth extension of T′.

QED

Thus, we have established

Conclusion 2.

There exist non-smooth extensions with languages of the same cardinality.

But, again one might object to our example.

Objection 2.

The original theory T′ in example 2 is not finitely axiomatizable (see, e.g. [Enderton '72, p. 184 (exercise 6)]).

3.3. Finitely axiomatizable T′

We now face objection 2 raised above by investigating non-smooth extensions where the original theory T′ is finitely axiomatized, and both languages have the same cardinality.

Example 3.

Consider the structure **M** of the natural numbers with zero and successor with its usual ordering in an appropriate language L′ with *0*, *S* and <.

Let T′ = Th(**M**). Notice that it is finitely axiomatizable (see, e.g. [Enderton '72, p. 178, 187 (corollary 32B)] .

The set of axioms to be added is defined as follows. We use once more the variable-free terms *n* for each natural n. Consider a new constant *c*, let L″ = L′ ∪ {*c*}, set D = {*n* < *c* / n ∈ N} and T″ = T′ ∪ D.

Again, we have an instance of a general phenomenon dealt with in the next proposition, whose proof will establish that theory T″ = T′ ∪ D is a non-smooth extension of T′. For this purpose, it is convenient to introduce some notation. For each natural n, consider the sentence

$$n\text{-}pred : \exists x_1 \exists x_2 \dots \exists x_n [x_1 < x_2 \wedge \dots \wedge x_n < c]$$

We shall denote by Infprd(L′) the set of all such sentences n-pred. Notice that each sentence n-pred is a consequence of T″ of our example 3. We can now turn to our generalization of this example.

Proposition 3.

Any theory T with an infinite model has some complete extension T′ that has non-smooth extensions.

Proof.

By Löwenheim-Skolem theorem, consider a countably infinite model **A** of T; so that there is a bijection h between its domain A and the set N of the naturals. Add a new binary predicate symbol *p* to the language to form L′ and use the bijection h to induce the ordering < of N on A as a realization for *p*. This gives an expansion A′ of A so that h is an isomorphism between the structures < A, **A′** [*p*] > and <N, < >. Take T′ = Th(A′).

Pick a new constant *c* and add it to L′ to form L″.

For each natural n, consider the following sentence of L″

$$n\text{-}pred : \exists x_1 \exists x_2 \dots \exists x_n [p(x_1, x_2) \wedge \dots \wedge p(x_n, c)].$$

We shall denote by Infprd(L′) the set of all such sentences n-pred, and set T″ = T′ ∪ Infprd(L′).

Notice that model A′ of T′ cannot be expanded to a model of T″ in view of the ordering of N.
On the other hand, T″ is consistent, as can easily be seen by compactness (any finite subset of Infprd(L′) can be satisfied in some appropriate expansion of **A′**).
Therefore, the basic lemma yields that T″ is a non-smooth extension of T′.
QED

We can now state

Conclusion 3.
There exists a non-smooth extension of a finitely axiomatizable theory without increasing the cardinality of the language.

But, one can still be dissatisfied with our example.

Objection 3.
The new symbols appear in infinitely many new axioms.

3.4. Locally finite extension

Let us now face objection 3 raised above by showing how to deal with the case of new symbols appearing in infinitely many new axioms, i. e. when it is not locally finite in the following sense.

Call an extension *locally finite* iff each new symbol occurs only in finitely many new axioms.

Example 4.
Consider again theory T′ presented in example 2. Pick a new constant c_n, for each natural n, and add them to L′ to form the new language Finloc(L′).
Now, consider the set Finloc(Unnam(L′)), consisting of
— the new versions of the axioms of Unnam(L′) : $\neg\, c_n \simeq n$,
— the new equality axioms : $c_n \simeq c_{n+1}$,
for every natural n.
Notice that each new constant c_n occurs in at most 3 new axioms of Finloc(Unnam(L′)).
Also, among the consequences of Finloc(Unnam(L′)), one finds $\neg\, c_0 \simeq n$ and $c_n \simeq c_0$, for every natural n.

Example 4 shows a finitely localized version of the extension presented in example 2. But, one may very well complain that theory T′ in examples 2 and 4 is not finitely axiomatizable. To this we reply that we can apply the same trick to example 3. Indeed, we have here an example of a general construction, which we shall now describe and then show that it preserves both conservativeness and expansiveness.

Consider an extension T″ = T′ ∪ D, where the language L″ of T″ is obtained from the language L′ of T′ by adding a single new constant c. Choose a new constant c_n, for each n ∈ N, and add them to L′ to form the new language Finloc(L′). Now, enumerate the sentences of D as F_n, for n ∈ N, and consider the set Finloc(D), consisting of the following sentences of Finloc(L′)

— the new equality axioms : $c_n \approx c_{n+1}$, for $n \in$ N, and $c_0 \approx c$;

— the new versions of the axioms of D : $F_n(c_n)$, for n ∈ N

(here $F_n(c_n)$ is the result of replacing every occurrence of c in F_n by c_n).

Finally, call Finloc(T″) = T′ ∪ Finloc(D) the *finite localization* of the extension T″ = T′ ∪ D.

Proposition 4.

Consider an extension T″ = T′ ∪ D.

(a) Finloc(T″) is finitely localized.

(b) Finloc(D) is equivalent to an extension by definitions of the set D of new axioms.

(c) The extension T′ ⊆ T″ is conservative (resp. expansive) iff the extension T′ ⊆ Finloc(T″) is so.

Proof.

(a) Each new symbol c_n occurs only in two new equality axioms and perhaps in $F_n(c_n)$.

(b) Let Def consist of the sentences $c_n \approx c$, which define c_n in terms of c, for n ∈ N. Thus, D ∪ Def is an extension by definitions of D, which is easily seen to be equivalent to Finloc(D).

(c) In view of (b), the extension from D to Finloc(D) is both conservative and expansive. Hence the lemma in section 2 yields the desired conclusions.

QED

Now, we can finitely localize the extension in example 3; then proposition 4 yields

Conclusion 4.

There exists a non-smooth, locally finite extension of a finitely axiomatizable theory without increasing the cardinality of the language.

But, one can still be less than happy.

Objection 4.

The original theory T′ is complete.

3.5. T′ not complete

Let us now deal with objection 4 raised above by examining the case of extensions of a perhaps incomplete theory T′.

Example 5.

Consider the language L′ with *0* and *S* of example 2. As axioms for theory T′ we take the following two sentences

$$\forall x \neg Sx \simeq 0 \text{ and } \forall x \forall y (Sx \simeq Sy \rightarrow x \simeq y).$$

Notice that theory T′ is incomplete; since, for instance, neither $\forall x \neg Sx \simeq x$ nor its negation is a consequence of T′ (see, e.g. [Enderton '72, p. 178, 184 (exercise 6)]). Now, pick a new constant *c* and let L″ = L′ ∪ {*c*} . As in example 2, we set T″ = T′ ∪ Unnam(L′), where as before, Unnam(L′) = $\{\neg c \simeq n \ / \ n \in N\}$.

Again, this example is an instance of a general phenomenon, dealt with in the next proposition, whose proof will guarantee that T″ = T′ ∪ Unnam(L′) is a non-smooth extension of T′.

Proposition 5.

Any theory T′ with a named model that only has infinite models has some non-smooth extension.

Proof.

As in the proof of proposition 2, pick a new constant *c* and let L″ = L′ ∪ {*c*}. Now, set T″ = T′ ∪ Unnam(L′), where once more, Unnam(L′) = $\{\neg c \simeq t \ / \ t \in \text{Nam}(L')\}$.

Consider an infinite named model A of T′. Notice that it cannot be expanded to a model of T″, because it is named.

It remains to show conservativeness. For this purpose, consider a sentence F of L′ and assume F ∈ Cn(T″).

Then, by compactness, $T' \cup \{\neg c \simeq t_1 \wedge \ldots \wedge \neg c \simeq t_n\} \models F$, for some finite $\{t_1, \ldots, t_n\} \subseteq \text{Nam}(L')$. But then, (see, e. g., the rule EI in [Enderton '72, p. 117, corollary 24H]) we have $T' \cup \{\exists x (\neg x \simeq t_1 \wedge \ldots \wedge \neg x \simeq t_n)\} \models F$. On the other hand, since every model of T′ must be infinite, we already have $T' \models \exists x (\neg x \simeq t_1 \wedge \ldots \wedge \neg x \simeq t_n)$.

Therefore, T′ ⊨ F, as well.

QED

Thus, we have established

Conclusion 5.

There exists a non-smooth, locally finite extension of an incomplete finitely axiomatizable theory without increasing the cardinality of the language.

Our examples and results have a flavor of infinity. Indeed, two recurring features have been

— the original theory T′ has (only) infinite models, and

— the extensions T″ have infinitely many new axioms.

We shall examine them in the sequel; the first one in the next section, and the second one in section 6.

4. Model-theoretic properties of conservative extensions

This section deals with some model-theoretic properties of conservative extensions. We first present a characterization of conservative extensions, which is slightly simpler than the usual one. We then use this characterization to examine the behavior of finite models under a conservative extension, showing that they always expand. This will clarify why our examples in the preceding section involved only infinite models.

We first recall some simple definitions and known results.

Consider structures A and **B** for language L. We say that A is *elementarily equivalent* to **B** (denoted by A ≡ **B**) iff for every sentence F of L, A ⊨ F iff **B** ⊨ F. We call A an *elementary substructure* of **B** (denoted by A ≤ **B**), or **B** an *elementary extension* of A, iff A ⊆ **B** and for every assignment s of variables into A, whenever A ⊨ F [s] then **B** ⊨ F [s] as well. The following well-known connection between these concepts (see, e. g., [Shoenfield '67, p. 74; van Dalen '89, p. 125]) will be used in the sequel.

Remark. *Elementary substructure implies elementary equivalence*

If A is an elementary substructure of **B** then A is elementarily equivalent to **B**.

We can now present some model-theoretic characterizations of conservative extensions, which generalize the model-expansion criterion in section 2.

Theorem. *Model-theoretic characterizations of conservativeness*

For an extension T′ ⊆ T″ the following are equivalent.

(a) The extension T′ ⊆ T″ is conservative.

(b) Every model A of T′ has an elementary extension **B** that can be expanded to a model C of T″.

(c) Every model A of T′ is elementarily equivalent to a structure **B** that can be expanded to a model C of T″.

Proof.

(a ⇒ b) See [Shoenfield '67, p. 95, 96, (exercise 9.c)] .

(b ⇒ c) Clear, in view of the preceding remark.

(c ⇒ a) Consider a sentence F of L′ in Cn(T″). Let A be a model of T′. So,

by (c), there exists some model **C** of T″ such that **C** | L′ ≡ **A**. Then, **C** ⊨ F with F in L′, hence **C** | L′ ⊨ F and **A** ⊨ F as well. Therefore F ∈ Cn(T′).
QED

The above characterizations are generally not too easy to use in practice. First, (b) is not very easily applicable because it can be hard to establish that **A** ≤ **B**, since this involves practically all formulas of L′. In this respect, (c) is slightly simpler, in that checking elementary equivalence involves only sentences of L′, but still all of them. Even though there are some model-theoretic characterizations of elementary equivalence, one still has to face the task of producing such a **B** from any given **A**, in either case.

We now use this characterization to examine the behavior of finite models under a conservative extension, showing that they always expand. This will be based on the following well-known lemma (see, e. g., [Enderton '72, p. 96 (exercise 17)]).

Lemma. *Elementary equivalence with finiteness implies isomorphism*
If **A** is elementarily equivalent to **B** and **A** is finite, then **A** is isomorphic to **B**.

Proposition. *Expansion of finite models*
If T″ is a conservative extension of T′, then any finite model of T′ can be expanded to a model of T″.
Proof.
Let **A** be a finite model of T′. Then, by part (c) of the above characterizing theorem, there exists some model **C** of T″ such that **C** | L′ ≡ **A**. Since **A** is finite, by the preceding lemma, it is isomorphic to **C** | L′. Hence, there exists **D** isomorphic to **C**, and so **D** ⊨ T″, such that **D** | L′ = **A**.
QED

The next immediate corollary explains why our examples of non-smooth extension in the preceding section involved theories with only infinite models.

Corollary.
If T′ only has finite models, then any conservative extension of T′ is expansive as well.

5. Characterizing conservativeness

In this section we shall reexamine some of the examples of non-smooth extension presented in section 3. This will suggest an explicit characterization of non-smooth extensions in terms of the axioms added. This, in turn, will provide a new characterization for conservative extensions by the addition of constants, which is more easily applicable.

Let us reexamine the examples of section 3 involving the addition of a single new constant, namely examples 2, 3 and 5.

First, let us consider examples 3.2 and 3.5, with *0* and *S*.

The set D of axioms added to T′ consisted of the sentences of the form F(*c*), where F(*x*) is the formula $\neg x \simeq n$, for each variable-free term *n*. A sentence $\exists x$ G(*x*), where G(*x*) is a conjunction of some formulas F(*x*)'s, asserts the existence of an element distinct from those mentioned.

Now, let us examine example 3.3, involving *0*, *S* and <.

The set D of axioms added to T′ consisted of the sentences of the form F(*c*), where F(*x*) is the formula $n < x$, for each variable-free term *n*. A sentence $\exists x$ G(*x*), where G(*x*) is a conjunction of some formulas F(*x*)'s, asserts the existence of an element greater than those mentioned.

Notice that the following remarks apply to these three examples. First, any sentence $\exists x$ G(*x*) as above is in Cn(T′) (a fact that was actually exploited in the proof of proposition 5 in section 3 in order to establish the conservativeness of the extension in example 3.5). Second, in the (standard) model presented one cannot jointly satisfy all the formulas F(*x*).

We shall now show how these ideas can be generalized to provide an explicit characterization of non-smoothness, as well as a better characterization for conservativeness, for extensions by the addition of constants.

Consider a theory T′ over language L′. Let L″ be obtained form L′ by the addition of some new symbols and let T″ = T′ ∪ D, where D is a set of sentences of L″. We call D *rough* with respect to T′ iff the extension T′ ⊊ T″ is non-smooth.

We consider first the case where a single new constant is added to L′ to form L″.

By means of an appropriate alphabetic variant, if necessary, we may assume that variable *x* does not occur in any sentence of the set D of new axioms. For each sentence F of L″, let F(*x*) be the result of replacing every occurrence of *c* in F by the new variable *x*. (Notice that if *c* does not occur in F, then F(*x*) is F and $\exists x$ F(*x*) is equivalent to F itself.) Now, form the set D(*x*) = {F(*x*) / F ∈ D} . Let &D(*x*) be the set of all finite conjunctions of formulas of D(*x*), and, finally, let $\exists x$D be the set {$\exists x$ G / G ∈ &D(*x*)} of sentences of L′.

We call D *unlimited* with respect to T′ iff every sentence in $\exists x$D is a consequence of T′, but there exists some model **A′** of T′ where the set D(*x*) is not jointly satisfiable.

The following lemma is immediate.

Lemma. *Jointly satisfiability vs. expandability*

The set D(*x*) is jointly satisfiable in a structure **A′** for L′ iff **A′** can be expanded to a model **A″** of D.

We can now provide our explicit characterization of non-smooth extensions, by characterizing rough sets of new axioms.

Proposition. *Roughness vs. unlimitedness (single new constant)*
A set D is rough with respect to T' iff D is unlimited with respect to T'.
Proof.
Given the preceding lemma, it suffices to verify that conservativeness is equivalent to "preservation of conjunctions".
($\Rightarrow$) Given a finite conjunction $G(x)$ in $\&D(x)$, $G(c) \in Cn(D)$. Thus, $\exists x\, G(x)$ is a sentence of L' in Cn(T''), whence it is in Cn(T'), by conservativeness.
($\Leftarrow$) Consider a sentence H of L' in Cn(T''). By compactness, $H \in Cn(T' \cup \{G\})$, for some finite conjunction G of sentences of D. Then, (see, e. g., the rule EI in [Enderton '72, p. 117, corollary 24H]) $T' \cup \{\exists x\, G(x)\} \vDash H$. But $G(x) \in \&D(x)$, hence $\exists x\, G(x) \in \exists x D$. Thus, $T' \vDash \exists x\, G(x)$, whence $T' \vDash H$.
QED

We now indicate how to generalize the preceding considerations to the case where a set C (at most countable) of new constants is added to L' to form L''.

By means of an appropriate alphabetic variant, if necessary, we may assume that no variable x_{2n} with even index occurs in any sentence of D. Now, enumerate the set C of new constants as c_k, for $k \in N$, and consider the substitution s that replaces each new constant c_k by variable x_{2k}. Let s(F) denote the result of applying substitution s to formula F of L'', and consider the set $s(D) = \{s(F) / F \in D\}$. Let &s(D) be the set of all finite conjunctions of formulas of s(D) and let $\exists s(D)$ consist of the existential closures of the sentences of &s(D).

We call D *unlimited* with respect to T' iff every sentence in $\exists s(D)$ is a consequence of T', but there exists a model **A'** of T' where the set s(D) is not jointly satisfiable.

We again have that s(D) is jointly satisfiable in a structure **A'** for L' iff **A'** can be expanded to a model **A''** of D. We can now see that our previous arguments extend to this case, thereby providing our explicit characterization of non-smooth extensions, by characterizing rough sets of new axioms, involving at most countably many new constants.

Theorem. *Roughness vs. unlimitedness*
Let language L'' be obtained form L' by the addition of an at most countable set C of new constants. Given a set D of sentences of L'', D is rough with respect to T' iff D is unlimited with respect to T'.

We now have our simpler characterization of conservative extension by the addition of new constants, which is more easily applicable.

Theorem. *Conservativeness vs. expansiveness and unlimitedness*

Let L″ be obtained from L′ by the addition of an at most countable set C of new constants. Consider a theory T′ over L′ and its extension T″ = T′ ∪ D, where D is a set of sentences of L″. Then, the extension T′ ⊆ T″ is conservative iff
either it is expansive,
or else, D is unlimited with respect to T′.

The usefulness of this theorem in checking conservativeness stems form its simplicity. It tells us that when the simple model-theoretic criterion of section 2 fails, then conservativeness will be equivalent to an explicit property of the set of new axioms.

6. Finite conservative extensions

All the examples of conservative, but non-expansive, extensions in section 3 involved the addition of infinitely many new axioms. In section 5 we have characterized the conservative extensions by the addition of constants. So, the question that remains concerns conservative extensions with only finitely many new axioms. In this section, we will show that such extensions are always expansive.

We shall be using the idea of Skolem constant, a simple special case of Skolem function (see, e.g. [van Dalen '89, p. 144-146]).

Consider a formula F of L′ of the form $F(y_1, ..., y_n)$, with no variables other than the displayed ones occurring free in it. Let language L″ be obtained from L′ by the addition of some new constants, $c_1, ..., c_n$ being among them. Replace every free occurrence of each variable y_k in $F(y_1, ..., y_n)$ by the new constant c_k; this yields the sentence $F(c_1, ..., c_n)$ of L″. This sentence $F(c_1, ..., c_n)$ is called a *Skolemization* of the sentence $\exists y_1 ... \exists y_n$ F of L′.

Now, consider theories T′, over L′, and T″, over L″. We call T″ a *Skolem extension* of T′ iff every new axiom of T″ is a Skolemization of some consequence of T′. Notice that, in this case, every new constant of L″ occurs in a single new axiom of T″, called its constraining axiom.

The following lemma recalls a simple result concerning these concepts. Its proof is simpler than its well-known counterpart for Skolem functions, in that it does not require the Axiom of Choice.

Lemma. *Expansiveness and conservativeness of Skolem extensions*

If T″ is a Skolem extension of T′, then
(i) T″ is an expansive extension of T′,
(ii) T″ is a conservative extension of T′.

Proof.

(i) Consider a model **A′** of T′.
Let $F(c_1, ..., c_n)$ be a new axiom of T″. It is a Skolemization of $\exists y_1 ... \exists y_n F(y_1, ..., y_n)$, which is a consequence of T′. Thus, we know that $F(y_1, ..., y_n)$ is satisfiable in **A′**. Thus, by the lemma in section 5, **A′** can be expanded to a structure which is a model of this new axiom.
Since each new constant occurs in a single new axiom, we can thus expand **A′** to a model **A″** of T″.
(ii) follows from (i) in view of the model-expansion criterion in section 2 (it is also a special case of the theorem on functional extensions [Shoenfield '67, p. 55]).
QED

We can now characterize the conservative extensions by a set of new constants formed by the addition of only finitely many new axioms.

Theorem. *Finite conservative, expansive and Skolem extensions*
Consider a language L′ and let L″ be obtained from L′ by the addition of a set C of new constants. Consider a theory T′ over L′ and its extension T″ over L″ obtained by adding to T′ a finite set D of sentences of L″. Then, the following are equivalent.
(a) The extension $T' \subseteq T''$ is conservative.
(b) The extension $T' \subseteq T''$ is equivalent to a Skolem extension.
(c) The extension $T' \subseteq T''$ is expansive.
Proof.
(a ⇒ b) Let F be the conjunction of the new axioms in D. Clearly, $T' \cup \{F\}$ is equivalent to T″. We shall now show that $T' \cup \{F\}$ is equivalent to a Skolem extension of T′. First, F has only finitely many new constants, say n of them. By resorting to an alphabetic variant, if necessary, we may assume that variables $y_1, ..., y_n$ do not occur in F. Now, let F^* be obtained from F by replacing each new constant by a corresponding new variable y_k. Then, F will turn out to be equivalent to a Skolemization of $\exists y_1 ... \exists y_n F^*$. But, the latter is a sentence of L′ in Cn(T″), whence in Cn(T′), by conservativeness.
(b ⇒ c) By the preceding lemma.
(c ⇒ a) By the model-expansion criterion in section 2.
QED

The above theorem shows that, for extensions by constants with only finitely many new axioms, conservativeness and expansiveness are equivalent, because all such extensions turn out to be Skolem extensions. This has some interesting consequences. First, it clarifies why our examples of non-smooth extensions in section 3 involved the addition of infinitely many new axioms. Second, it shows that Skolem extensions, in addition to being conservative, turn out to be the most general conservative extensions by addition of constants.

7. Conclusions

We have examined conservative and expansive extensions, emphasizing extensions by constants. The causes of conservativeness without expansiveness have been illustrated and examined, and then conservative extensions by constants have been characterized. Finally, the finite conservative extensions by constants have been characterized as the Skolem extensions, which are always expansive.

The interest in conservative extension is due to its importance both in mathematical logic and in formal specifications. In the former, they are useful, for instance, in proving relative consistency [Enderton '72; Shoenfield '67; Smirnov '86], and also because they encompass several useful generalizations of extensions by definitions [Veloso + Veloso '90] . In the latter, they are used, for instance in program and specification development by stepwise refinement [Ehrig + Mahr '85; Turski + Maibaum '87; Maibaum + Veloso + Sadler '84; Veloso '87], as well as in formalizing some useful intuitive ideas [Polya '57] concerning problems, solutions, analogy and problem-solving methods [Veloso + Veloso '81; Veloso '84; Veloso '88] .

We have started in section 2 by reviewing some usual terminology, notation and results of mathematical logic and introducing the concept of expansive extension. Section 3 has the character of a « Socratic » dialogue, in that we have examined the causes of non-expansiveness by presenting and illustrating a series of ways of obtaining non-smooth (i. e., conservative but not expansive) extensions, each one of them being improved by facing the objections raised against the preceding one. The conclusion of this process is that the possibility of obtaining non-smooth extensions seems to stem from some « infinite character » of the original theory.

In section 4 we have characterized conservative extensions by means of model-theoretic properties, which have been useful in investigating the behavior of finite models. Then, in section 5 we have provided a simple characterization for the conservative extensions by constants, by showing that they are either expansive or the axioms added are unlimited (the existential closures of a translation of their finite conjunctions are consequences of the original theory, but this theory has a model where the set of new axioms cannot be jointly satisfied). Then, in section 6 we have characterized the finite conservative extensions by constants as those obtained by the addition of Skolem constants, which are expansive.

These results provide some interesting conclusions. First, the finite models can always be expanded if the extension is conservative, and a conservative extension by constants with only finitely many new axioms is always expansive. Thus, the only way to obtain non-smooth extensions by constants is by the addition of infinitely many new axioms to a theory with infinite models. This settles the question hinted at in section 3, by corroborating the feeling that the possibility of obtaining non-smooth extensions hinges on some « infinite character » of the original theory. In section 3, we have illustrated some typical ways of obtaining such extensions.

Another interesting conclusion to be drawn from our results concerns Skolem extensions. We have shown in section 6 that any extension by constants with finitely many new axioms is equivalent to a Skolem extension, in that it amounts to the addition of Skolem constants corresponding to consequences of the original theory. Thus, Skolem extensions, in addition to being always expansive, provide the most general way of obtaining finite conservative extensions by constants.

We have emphasized extensions by the addition of constants, because we wished to stress the existence of non-smooth extensions even under very severe restrictions. But, some of our considerations and results carry over, with appropriate adaptations, to extensions by function symbols. For instance, the above conclusion that Skolem extensions, in addition to being expansive, are the most general finite conservative extensions still applies to some extensions by functions symbols. One simple, but interesting, such case occurs when the new axiom has the form $\forall\ x\ F(x, f(x))$, $F(x, y)$ being a formula of L′ where the term $f(x)$ is substitutable for y, with x being a list of variables $x_1, ..., x_n$ [Veloso + Veloso '90] .

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