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Proofs, what they prove and their representations: remarks in connection with identity of proofs and the normalisation thesis

Provas, o que elas provam e suas representações: observações em conexão com a identidade de provas e a tese da normalização

Abstract

A topic of de Castro Alves (2019) stands in need of re-visitation, namely: possible ways of specifying restrictions on the notion of proof and some other related ones that are relevant in connection with the discussion on identity of proofs. This effort is dedicated to start a compensation for relevant shortcomings of some ideas proposed in that work. More concretely: by taking some very generic traits of proofs as a departure point, we proceed to the identification of possible outset conditions *upon the investigation of identity of proofs (instead of proposing a taxonomy of criteria of identity of proofs, as in de Castro Alves (2019)). We will describe and briefly comment on two kinds of such conditions: one given in terms of how the identity of proofs is conditioned by the identity of what is proved, and other in terms of how equivalence relations between proof (re)presentations are conditioned by, on the one hand, how many distinct (collections/kinds of) proofs can be (re)presented by them, and, on the other, how many distinct (re)presentations a collection of proofs may have. To exemplify the meaningfulness of these considerations, they will be used here as basis for some critical remarks on the normalisation thesis on identity of proofs.*

Keywords: proofs; results; presentations; representations; identity

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Resumo

Um tópico de de Castro Alves (2019) precisa ser revistado, a saber: possíveis maneiras de especificar restrições à noção de prova e outras relacionadas que são relevantes em conexão com a discussão sobre identidade de provas. Este esforço inicia uma compensação por insuficiências de algumas ideias ali propostas. Concretamente: tomando traços bastante genéricos de provas como ponto de partida, procede-se à identificação de possíveis condições de saída *sobre a investigação da identidade de provas (ao invés de propor uma taxonomia de critérios de identidade de provas, como em de Castro Alves (2019)). Dois tipos de tais condições serão descritos e comentados: um formulado em termos de como a identidade de provas é condicionada pela identidade do que é provado; e outro em termos de como relações de equivalência entre representações/ apresentações de provas são condicionadas por, de um lado, quantas (coleções/ tipos de) provas podem ser representadas/ apresentadas por elas, e, de outro, quantas representações/ apresentações distintas uma (coleção/tipo de) prova pode ter. Para exemplificar a significatividade dessas considerações, elas servirão aqui de base para algumas observações críticas a respeito da tese da normalização sobre a identidade de provas.*

Palavras-chave: provas; resultados; apresentações; representações; identidade.

Introduction

One of the main obstacles to be faced by attempts at discussing proofs philosophically is to give a sufficiently clear specification of what one is going to talk about without making the discussion uninteresting for a (too) significant portion of the potential audience. I will thus not start the investigation by setting methodological standards that will *a priori* dismiss considerations on certain – obviously relevant – subject-matters referred to under the label "proof" as somehow impertinent to the present goals; like e.g. the Frege of the *Grundlagen* quite violently does with "psychologistic" notions in the opening of his contribution to the different (yet in many respects similar) discussion on numbers. This kind of move is most frequently peremptory and unwarranted; when not, it would in any case demand some very extensive underpinning that is not my intention to try to provide. That said, my attempt will be to circumvent, rather than solve this issue, by letting the investigation be guided by certain aspects of proofs that, at least in some sense and to a significant extent, arguably do not hinge upon such specifications. But let me first depict the issue at stake more concretely.

When discussions that concern the question "What is a proof?" get started, it is usual that a specification is given of what sort of "entity" is to be the object of scrutiny.[1] Thus: by "proofs", we may refer to ontologically independent abstract objects; to certain kinds of mental or linguistic acts, performances or processes; to the product of such acts, performances or processes; to certain kinds of linguistic expression; to the syntactic or semantic side of such expressions; to certain kinds of socio-historically determined events; to mental or linguistic (re)presentations of any of these things; etc. Notice that not only there is no clear guideline as to what should be included or excluded from this list of possibilities, but also the categories in it are not necessarily disjoint. Thus, certain kinds of linguistic expression are to be understood precisely as socio-historically determined events – and taking precisely this into account may be seen as crucial for e.g. a proper comprehension of at least some of their central semantic aspects – say, in a rather wittgensteinian, "no-vacation-for-language" kind of spirit. And notice: with respect to the matter of identity of proofs – the question that shall be the axis of the considerations to be made throughout this effort –different such specifications may drastically alter what one should be willing to accept as adequate criteria for identifying or distinguishing proofs. So, in a setting full of so many and sometimes so complexly connected options, what criterion should one use to pick one among these possibilities?

On this point, it is always good to remember that dealing with the question "what is a proof?" need not involve anything similar to an epic "τί ἐστι" quest for *THE* notion of proof, the nature of which is lying in the deep shadows of oblivion, waiting to be discovered and revealed by some chivalrous εἰδότα φῶτα. As far as we are concerned, there is actually no reason to assume that such a quest would have any point at all. To a large extent, we maintain the stance expressed in de Castro Alves 2019: we use proofs and talk about them in various contexts; and both our use and talk of proofs should provide

1 Chapter 11 of Novaes 2020, for instance, is a perfect example.

us with enough indication to, in each proper context, properly carry out the evaluation of any attempted answer to questions regarding what proofs are. Generality does not have to be an aim.

That said, the strategy to be employed in this effort is not to choose any among these possibilities, nor to seek any criteria that could justify such a choice. Rather, the idea is to provide means to see these possible ways of conceiving of proofs as components of a cohesive notion – fibers of a same rope, to use yet again a wittgensteinian image –, in spite of the fact that they are sometimes so different or even disconnected from one another, and to do that *without* taking any of them as somehow more fundamental or central than the others.

The first of the two key traits that will set the limits to what we call proofs in the present context is:

(a) proofs are always proofs *of* something.

The second of the two key traits should find optimal opportunity to be discussed after our considerations concerning (a):

(b) proofs can be (re)presented.

As remarked just above, the departure hypothesis is that these traits of proofs do not hinge upon the mentioned kind of circumscription usually in force from the very start of investigations concerning proofs. If one is not willing to accept these traits as faithful descriptions of proofs within the scope addressed by the present considerations, one may just take them as definitions of this scope without too much prejudice to the points to be made. In the immediate sequel, we will explore trait (a) further in connection with identity of proofs.

a. Proofs are always proofs of something

This transitive structure of proofs seems pervasive and incontrovertible enough: every proof inevitably brings with itself also *what it proves* – which we shall henceforth call the *result* of the proof.

In de Castro Alves 2019, it was claimed that, depending on how one chooses to deal with the relationship between the identity of a proof and the identity of its result, accordingly different *criteria* of identity of proofs may be deemed adequate viz. inadequate. In spite of not being wrong as it stands there, this formulation does not facilitate the appreciation of the matters which it is intended to help clarifying, and may be misleading for a number of reasons. To avoid this shortcoming, instead of taxonomising *criteria* of identity of proofs, we will here list kinds of *outset conditions* upon criteria of identity of proofs based on the different possible understandings of how the identity of proofs and that of their results are related. By outset conditions we mean here certain presuppositions concerning the notion of identity of proofs assumed to be in force independently of which particular criterion of identity of proofs is advocated.

Let us start by considering something Došen 2003, p.14, puts concerning the field of *general proof theory*, i.e. the study of proofs for their own sake, with no *a priori* imposed restrictions of method[2] :

> *"For the whole field of general proof theory to make sense, and in particular for considering the question of identity criteria for proofs, we should not have that any two derivations with the same assumptions and conclusion are equivalent, i.e. it should not be the case that there is never more than one proof with given assumptions and a given conclusion. Otherwise, our field would be trivial. This marks the watershed between proof theory and the rest of logic, where one is not concerned with proofs, but at most with consequence relations. With relations, we either have a pair made of a collection of assumptions and a conclusion, or we do not have it. In proof theory, such pairs are indexed with various proofs, and there may be several proofs for a single pair."*

Došen refers here indirectly to a folkloric background conception, namely: what is relevant about proofs is ultimately what they prove, two proofs being thus equivalent if and only if they prove the same thing. The rejection of precisely this idea seems to have triggered the young literature dedicated to identity of proofs[3]. Of course, the question of just what a proof proves viz. what we understand to be the result of a proof is to be answered satisfactorily if any clarity is to come from such an attitude towards identity of proofs – a

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2 Cf. Prawitz 1971, pp.236-237.

3 Kreisel 1965, p.117 is the first explicit mention of the issue to my knowledge. Prawitz 1971 is one of the most important early references, where the normalisation thesis on identity of proofs is put forward (p.257) and Lambek 1972 is credited for the suggestion of what Došen calls "generality conjecture" on identity of proofs.

question mostly neglected by discussions within this young literature, by the way. In any case, this attitude towards identity of proofs is tantamount to reducing the question regarding the identity of a proof to one regarding the identity of whatever it is that we consider as the result of a proof; in other words, it is a trivialization of the identity of a proof with respect to that of its result. This would be, then, a way of accounting quite sufficiently for the identity of proofs while dismissing the question itself as ultimately uninteresting: to reduce the identity of a proof to that of its result, thus resolving the initial question in terms of one that does not necessarily have anything whatsoever to do with proofs. This brings as a consequence that the very idea of a general proof theory puts clear outset conditions upon one's understanding of identity of proofs: there must be not only the possibility to differentiate between proofs in general, but also between proofs of the same results. Thus, for partisans of this kind of enterprise, identity of proofs is assumed from the outset to be an in principle non-trivial relation not only in general – which is hardly unexpected –, but also when restricted to proofs of a *specific result*.

We shall return to that observation of Došen soon; for now, it suffices to note that it commits to the idea that the possibility of there being different proofs of at least some given result is a necessary condition for an adequate approach to the notion of proof.

The stance of general proof theory described by Došen implies a rupture with one of the directions of the precedent, folkloric one, but not with the other – i.e. proving the same result is not anymore considered a sufficient condition for two proofs to be equivalent, but it may well remain as a necessary condition for this much. And it seems to be precisely the decision to regard having the same result as a not sufficient yet necessary condition for two proofs to be equivalent that motivates most of the actual developments on identity of proofs available. This is indeed the first attitude towards identity of proofs that does not trivialize or dismiss the question in any sense, viz. the identity value of a proof is made neither trivial nor reducible to that of something else. Nevertheless, the identity of the result still plays a prominent role in the determination of the identity of the proof in this view, and an account of it must be provided so that the limitation it imposes upon the identity of the proof becomes clear.

One could still of course move a step further and conceive the possibility of proofs of different results being equivalent, which would imply denying also the other direction of the thesis criticized by general proof theory. This indeed hardly could be regarded as an extravagant hypothesis; for proofs of what could be considered different things might be strongly analogous in various and significant senses.[4] In such a framework, it is neither necessary nor sufficient that two proofs share the same result for them to be equivalent – i.e. there may be distinct proofs of one and the same result, and there may also be equal proofs of distinct results. In this conception, then, the identity of a proof is in principle neither trivialized, nor reduced to the identity of something else, nor restricted in any decisive way by the identity of its result.

These different possible ways of approaching identity of proofs – and, consequently, proofs – allow us to identify a list of outset conditions that might be imposed upon criteria for identity of proofs as specifications of how these are in principle related to the identity of proof results.

Let "≡" stand for the relation of identity of proofs and "=" for the identity of proof results. With respect to the relation between a proof and the result of a proof, the consideration of criteria for the identity of proofs may be conditioned by the following outset assumptions:

1) **Identity of results is a necessary condition for identity of proofs:** Given that Π is a proof of A and Σ is a proof of B, this condition establishes that, for any proposed interpretation of "≡", it must hold that, if $\Pi \equiv \Sigma$, then $A = B$. If this outset condition is in force, we say that the context of discussion of identity of proofs is *restricted*; otherwise, the context is called *unrestricted*.

   Remark: notice that restriction of context does not imply the non-triviality of "≡" – that would take the additional assumption that e.g. identity of proof results "=" is not trivial.

4 To illustrate the idea, the reader could consider two derivations of different end-formulas that nevertheless have the same generality – which would imply their equivalence according to the generality conjecture, mentioned in footnote 3. A concrete example is

$$\dfrac{\dfrac{A\wedge(A\supset B)}{A\supset B}}{(A\wedge(A\supset B))\supset(A\supset B)} \quad \text{and} \quad \dfrac{\dfrac{A\wedge(A\vee B)}{A\vee B}}{(A\wedge(A\vee B))\supset(A\vee B)}$$

The idea is roughly that the rules applied in each case are exactly the same, which arguably accounts for the fact that the difference between top and end-formulas is not detrimental of the fact that the very same inferential procedure (viz. proof) is displayed in both derivations. For further details on the generality conjecture, the reader is referred to section 3 of Došen 2003.

2) Identity of proof results is a sufficient condition for identity of proofs: Given that Π is a proof of A and Σ is a proof of B, this outset condition establishes that, for any kind of proposed interpretation of "≡", if A = B, then Π ≡ Σ. If this outset condition is in force, we say that the context of discussion of identity of proofs is *undiscriminating*; otherwise, it is said to be *discriminative*.

Remark: notice that only if discriminativeness obtains might it be the case that proofs of identical results are not identical – thus the terminology *discriminative*.

Let us now describe some properties that ≡ might display concerning the proofs it identifies. The relation of these properties with the outset conditions just described is to be explored right after:

3) Relative triviality: ≡ is relatively trivial iff, given that Π is a proof of A and Σ is a proof of B, Π ≡ Σ if and only if A = B.

4) Absolute triviality: ≡ is absolutely trivial iff, given that Π is a proof and Σ is a proof, then Π ≡ Σ.

5) Reverse triviality: ≡ is reversely trivial iff every proof Π is only identical to itself.

Now, depending on how the outset conditions 1 and 2 are combined, different limits are set beforehand as to which proofs can be identified by proposed identity criteria. Such limits are given precisely in terms of the properties 3, 4, and 5 above. This happens as shown in the following table:

How identity of proof results conditions identity of proofs			
At most \ at least		Sufficient condition?	
		Yes	No
Necessary condition?	Yes	At most relatively trivial\ At least relatively trivial	At most relatively trivial\ At least reversely trivial
	No	At most trivial\ At least relatively trivial	At most trivial\ At least reversely trivial

As one can see, the decision between an undiscriminating or a discriminative context determines a lower bound, so to speak, to which proofs can be identified by a proposed identity criterion, whereas the decision between a restricted or unrestricted context determines an upper bound thereto. The different combinations thus determine different ranges in which it is possible that a proposed identity criterion is deemed correct.

The point to be noted now is that the only combination that dispenses with the need of justification is "No – No": for it is the only one that does not block any possible relation candidate to the role of identity of proofs from the outset. Notice that this variant does not mean that identity of results in fact is neither a necessary nor a sufficient condition for the determination of the identity of proofs; it only states that nothing of this much is assumed from the outset to hold with respect to identity of proofs.[5] So, if we eventually come to the conclusion that e.g. only proofs of the same result are identical, this must not be an *a priori* limit to our investigation of which identity criteria for proofs are suitable, but should rather obtain, if at all, as an *outcome* of this investigation, and thus duly underpinned by proper arguments. All other possible combinations of outset conditions are in need of justification – i.e. quite unsurprisingly, one is expected to *show* why the possibilities that e.g. proofs of the same result are different or proofs of different results are the same should, if at all, be excluded from the outset. In spite of its lack of any *grandeur*, this remark is of the utmost importance for the adequate appreciation of important contributions to the debate on identity of proofs in the literature of general proof theory[6].

The importance of stressing and not losing sight of the fact that this list of conditions and properties is of an informal nature could never be

5 Making this and akin distinctions clear is probably the greatest advantage of the present approach over that of de Castro Alves 2019. The latter taxonomised criteria of identity of proofs *themselves* instead of their outset conditions, making it rather more difficult to explain such distinctions (e.g. it is *prima facie* difficult to see how the normalisation thesis can be interpreted as providing an unrestricted *criterion*, given that it actually *does not* identify any derivations that do not share both end- and undischarged top-formulas.

6 Of particular interest is the import of a strategy to argue for the completeness of Prawitz's normalisation thesis on identity of proofs by proving the maximality of the correspondent equivalence relation between derivations. This is especially true after the obtainment of exactly such maximality results concerning fragments of propositional logic in the early 2000s (cf., e.g., Došen and Petrić 2000, Došen and Petrić 2001, Widebäck 2001). This topic has been discussed in de Castro Alves 2019 and has a separate (still unpublished) article from this author dedicated to its discussion, which benefits significantly from the present exposition. It will only be very briefly mentioned here on section a.1., for an illustrative purpose.

overestimated, so this is to be stated now, before any confusion on this matter finds opportunity to come about: until this point, the expressions "proof" and "result" all have an essentially informal meaning; so "proof" does not mean derivation, "result" does not mean end-formula, etc. This sort of assimilation of meaning which we are blocking here is, by the way, precisely what seems to happen at the transcribed passage of Došen 2003 above: assumptions and conclusions of *derivations* are *formulas,* just as syntactical as these are; assumptions and conclusions of *proofs*, on the other hand, are informal viz. semantically loaded. Thus, e.g. the triviality (or non-triviality) of a relation of equivalence between *derivations* of a certain A from a certain Γ is neither a necessary nor a sufficient condition for the triviality viz. non-triviality of any relation of equivalence between proofs of a certain conclusion from given assumptions – unless, of course, one *shows* that there is a correspondence of a specific nature between, on the syntactical side, formulas, and, on the semantical side, results of proofs. In principle, it could well be the case that, for every A and Γ, all derivations of A from Γ were equivalent to one another, and yet there still were different proofs of a given result from given assumptions – just let the distinct formulas A and B express the same proof result and the distinct sets of formulas Γ and Δ express the same proof assumptions, and further let e.g. no derivation of A from Γ be equivalent to a derivation of B from Δ, and *voilà*. Since we have argued neither for nor against any kind of correspondence relation between formulas and proof results, and since it is also fairly usual to see some such correspondence being taken for granted in the literature, the proviso just made is justified. Precisely this kind of relation is to be scrutinised in section b.

a.1. An illustration: an argument for the completeness of the normalisation thesis

Now, the most paradigmatic proposal to deal with identity of proofs within general proof theory, the normalisation thesis[7], states that two derivations in standard natural deduction should be understood as representatives of the same proof if and only if they reduce to the same normal form.

It should be noticed that the normalisation thesis involves two distinct claims: one to soundness (the "if" part), and other to completeness (the "only

7 For an explanation of what the normalisation thesis consists in and some of its basic presuppositions, see de Castro Alves 2020 (in Portuguese).

if" part) w.r.t. the relation of identity of proofs it intends to characterise. The first of the claims seems solidly based on an arguably reasonable interpretation of the so-called inversion principle (cf. Prawitz 1965) and, more broadly, on a Prawitz-Dummett verificationist proof-theoretic approach to the meaning of the logical constants, and has been put forward quite confidently by its proponents; whereas the second one is frequently treated in the literature as a sort of Achilles heel of the normalisation thesis. This much is clear since Prawitz's mentioned 1971 formulation of it, which is followed by his explicit acknowledgement that, while "it seems evident" that "a proper reduction does not affect the identity of the proof represented", it is "more difficult to find facts that would support" this half of the "conjecture". He nevertheless refers us to Kreisel 1971 (cf. Kreisel 1971, p.165, footnote 20), published in the same volume, in which the latter attributes to Barendregt an idea that would be a way out of this situation: a proof of the eventual maximality of the notion of identity determined by the normalisation thesis. The core of the idea is that the maximality of an intended formalisation of the notion of identity of proofs would be a decisive argument for its completeness in case its soundness is granted.

To bring the following points to an adequate level of clarity and self-containment, it will be convenient to provide a precise enough explanation of what we are referring to when talking of maximality here. Thus, let ≡ be a relation of equivalence that holds between derivations. We first say that a relation ≡' is an (Γ, A)-extension of ≡ iff there are two derivations Π and Σ of A from Γ such that Π ≢ Σ and Π ≡' Σ. A relation ≡ is understood as maximal with respect to (Γ, A) iff its only (Γ, A)-extension is the trivial relation, i.e. the relation that makes all derivations of A from Γ equivalent. A relation ≡ is thus maximal in case it is maximal with respect to every (Γ, A).

The structure of the argument is roughly the following: Suppose that (i) a given relation ≡ is sound with respect to identity of proofs; (ii) ≡ is also maximal; and (iii) identity of proofs is not a trivial relation. Now suppose further, for absurdity, that (iv) ≡ is not complete with respect to identity of proofs. Then, by (i) and (iv), there would be more proofs of A from Γ identical than those already identified by ≡ – i.e. the set of proofs identified by ≡ would be a proper subset of the set of all identical proofs. But given (ii), this would in turn imply that identity of proofs is trivial. By (iii), however, this is absurd. Then we deny (iv) and admit that ≡ is complete.

Now, supported by this reasoning, the actual obtainment of maximality results concerning the identity relation corresponding to the normalisation thesis (cf. e.g. abstract of Widebäck 2001 and Došen 2003, p.14) (which amounts to an authorisation to affirm premiss (ii)) has been widely regarded as an until then missing stamp of approval on the normalisation thesis; more precisely, as a strong – in fact, the strongest available – argument of a technical nature in its favour.

There is, however, a clear problem with this line of reasoning that is directly related to the decision between restricted and unrestricted context just described. As exemplified in the passage of Došen 2003 quoted above, it is standard procedure in the discussions of general proof theory at stake here to conflate formulas (or sequents) and proof results in such a way that to each distinct formula corresponds exactly one respectively distinct proof result and vice-versa.

Bearing this in mind, one should notice that the conclusion drawn from premiss (ii) ("≡ is maximal") and "the set of proofs identified by ≡ is a proper subset of the set of all identical proofs" in the argument – namely, that identity of proofs is trivial – is only cogent in the presence of an additional, implicit assumption: that the relation of identity of proofs viz. equivalence between derivations is understood, from the very outset and independently of the normalisation thesis or any other attempted formalisation of it, as limited to identifying only derivations which share the same conclusion and undischarged hypotheses.[8] Given the conflation of formulas/sequents and proof results just mentioned, this clearly shows that a commitment to what we called a *restricted context* of discussion, the usual departure point of discussions of identity of proofs within general proof theory mentioned above, is in action at the argument; which in turn imposes upon any proposed criterion of equivalence of derivations – and, in particular, upon the one expressed by the normalisation thesis – an "*a priori*" restriction, namely: if sound, it could not even in principle determine that derivations with different conclusions or undischarged hypotheses – which would stand for proofs of different results – are equivalent.

But the fact is that the normalisation thesis alone does not imply any commitment to a restricted context. As a matter of fact, it could perfectly well be the case that it identifies only derivations that share the same conclusions as a mere *consequence* of how e.g. the notion of identity-preserving transformation alone should be understood. In this second case, it is fair to understand that the normalisation thesis could, in principle, have identified derivations that do not share conclusion or undischarged hypotheses; and that the fact that

8 For an example of an interpretation of identity of proofs that does not conform to this restriction, see footnote 4. For some interesting considerations on the relation between the normalisation thesis and this proposal, the reader is referred to Došen 2003.

it only identifies derivations that do share them is an "*a posteriori*" restriction upon the equivalence relation it determines – i.e. the context could perfectly well be understood to be *unrestricted*. But if one supposes that this is the case, then the maximality of the formal equivalence relation candidate to the role of counterpart of identity of proofs is completely irrelevant: for in such a case, there would be no reason to determine beforehand that derivations which do not share both end-formula and assumptions represent distinct proofs. Given that the maximality results involved in Barendregt's argument concern solely extensions of the equivalence relation yielded by the normalisation thesis which identify more proofs *of given A from given* Γ than this equivalence relation itself does, they become obviously insufficient as an argument to support the completeness of the latter with respect to a relation that may in principle identify derivations which do not share both end-formula and assumptions.

This shows that the argumentative strategy under scrutiny depends crucially, as already claimed, on the supposition of a *restricted* context. As already suggested, it is not only not necessary but also rather controversial – even though very usual – to impose this sort of outset condition upon the notion of identity of proofs. Hence, in the current absence of good arguments to sustain such a move, it has nothing but the idea of identity of proofs/ equivalence of derivations outlined by the normalisation thesis itself to motivate it. In such a case, however, the maximality argument involves circular reasoning, bearing no probative import whatsoever with respect to the completeness of the normalisation thesis. Otherwise, the burden lies upon the shoulders of those willing to maintain the maximality argument to provide foundations to the specific way of *a priori* restricting identity of proofs upon which it depends.

Given that the maximality argument is considered one of the strongest reasons in favour of the normalisation thesis, this brief subsection efficiently illustrates the usefulness of the kind of conceptual distinction described here by correcting the *prima facie* possibly tempting idea that it is somehow negligible or inconsequential.

b. Proof (re)presentations

The relation between proofs and their (re)presentations is vital. (Re)Presentations of proofs here have, keeping the same inclusive attitude, a broad understanding: very roughly, something possibly distinct from the proof, but not necessarily so, such that, either in fact or in principle, can e.g. display, or concretise, or express, or convey, or depict, or stand for, etc. the proof.

The role is frequently played by various things, from inductively generated formula trees to conversations between friends supplemented by some scribbling on scrap paper. Indeed, as noted here before, this relation sets a limit to the scope of this effort: we deliberately waiver any attempt at accounting for eventual proofs that cannot be (re)presented somehow. Several reasons could be given in support of this option: distinctive epistemic contours seem to shape whatever one should be willing to call a proof, which arguably should make something like "a proof that cannot be (re)presented" sound rather oxymoronic. But then again, since ancient times have people stumbled upon things that sounded (or perhaps indeed *were*) oxymoronic at first, but not for longer than just until concepts have been so transformed as to accommodate them into meaningful roles in their practices and/or according to their goals. Be that as it may, I would rather acknowledge the limitation of scope than risking an unwarranted generalisation that should in any case bring little to no profit if measured against the present purposes.

A matter of fact: general proof theoretic investigations have so far subjected themselves to at least the very same scope limitation. Mostly, this happens in a very specific way: the proofs addressed are all represented by natural deduction derivations. Prawitz 1971 (p.258) is a *locus classicus* of the stance:

> *"We have argued at length (…) for the claim that Gentzen's systems of natural deduction constitutes a characterization (…) of (different kinds of) first order proofs. We may summarize this claim in the thesis: Every first order positive, intuitionistic or classical proof can be represented in M, I, or C* [the respective Gentzen-style natural deduction systems]*, respectively."*

There are several different aspects to account for concerning the relation between a proof and what (re)presents it, whatever it is. The one that shall concern us now is, again, motivated by the axis of our discussion here: the question of identity of proofs. It would not be extravagant to say that this question seems to have gained the attention of the few who deal with it in the literature due to, among other things, the adoption of a basic departure point: there is no 1 to 1 correspondence between (kinds/collections of) proofs and their (re)presentations. The idea behind this claim is that, were there such a correspondence, there would be no point in asking when e.g. two distinct derivations represent the same proof – the exact phrasing Prawitz 1971 (p.257) gives to the question concerning identity of proofs and which endures as the most frequent approach to the matter within general proof theory. Whether this departure point is good or bad is not relevant for our present purposes. What matters is now this: if 1 to 1 correspondence between proofs and their (re)presentations indeed fails, then at least one of the following is true: (b.1) there is more than one proof (kind/collection) associated to at least one proof (re)presentation; or (b.2) there is more than one proof (re)presentation associated to at least one proof (kind/collection). For we already have excluded the possibility of proofs devoid of (re)presentations, and a proof (re)presentation that (re)presents no proof is a contradiction in terms. Let us explore these possibilities further.

b.1. More than one proof per (re)presentation

There are quite evident contexts in which it makes sense to say that a given proof (re)presentation may (re)present more than one distinct proof or collection/kind of proofs. Think of e.g. a natural deduction derivation of a theorem and how it can be looked upon as a surrogate for many distinct concrete ways of explaining the deductive steps involved to a group of students in a class (with or without certain graphic resources, using different notations, ordering lemmas and sub-arguments in different ways, etc.). This may happen in significantly different ways, though: generality, vagueness, ambiguity and other phenomena may be responsible for an eventual multiplicity of proofs being associated with some proof (re)presentation; and, of course, these differences may be reflected in drastically different consequences to the matter of when these (re)presentations (re)present the same proofs.

b.1.1. The case of the normalisation thesis on identity of proofs

Now, the normalisation thesis seems to depend on taking for granted that no such phenomena happen within its scope. As already mentioned, according to the thesis, two derivations in standard natural deduction should be understood as representatives of the same proof if and only if they reduce to the same normal form. Arguably, the normalisation procedure is looked upon as an identity-preserving evaluation of the derivation, which, by freeing it from "detours" – the redexes –, would bring them into normal forms – which are in turn understood as direct, canonical representatives of the identity values in question.

This is where the uniqueness of normal form shows its importance. There are no two different ways of completely freeing a derivation from redexes such that they split it into two different normal versions of itself; every reduction sequence of a derivation that terminates does so leading to the very same normal form in the end. The fact that strong normalisation also holds, i.e. that every sequence of reduction terminates, makes the picture even more appealing: for there is then, additionally, no way of normalising a derivation that may not lead to a normal form. To sum up, every possible way of freeing a derivation from its irrelevant features – which is by necessity built exclusively out of identity-preserving transformations – leads to a same and unique redex-free version of itself. This version – its normal form – is thus fit for the role of a canonical "representative" of the identity values of the derivations which reduce to it, which are thus of course, *given the assumption of the non-ambiguity of the identity-value of derivations in general (and of normal derivations in particular)*, deemed to be the same.

But as innocent as this assumption may seem, it really stands upon thin ice in this discussion. Let us, for instance, consider the following derivation:

$$\dfrac{\dfrac{\bot}{A \vee B} \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}$$

Since it introduces a complex formula and eliminates it immediately after, one might look upon it as displaying a structure that fits the idea of a "detour" involved in the normalisation thesis (Indeed, the occurrence of $A \vee B$ is a maximum formula according to the definition given by Prawitz 1965, p.34). There would be, however, two obvious ways in which we could get rid of this:

$$\begin{matrix}\dfrac{\bot}{[A]}\\ \vdots \\ C\end{matrix} \quad \text{or} \quad \begin{matrix}\dfrac{\bot}{[B]}\\ \vdots \\ C\end{matrix}$$

Letting ourselves be guided by the motivating ideas of the normalisation thesis, it seems quite sensible to see these two simplified derivations as significantly different from each other – at the very least, in any case, their normalisation sequences clearly do not converge. This failure of uniqueness of normal form could be taken to point at a case of *ambiguity* of a derivation with respect to its identity value. This sort of phenomenon does not necessarily require the presence of detours; indeed, it is and that seem to be the main ingredients in these cases. For instance, consider the derivation:

$$\frac{\bot}{A \vee B}$$

Motivated by the ideal of separation of the roles of the logical constants and the simplicity of inference rules (as e.g. described by Prawitz 1971, section II.2.1), one could regard this derivation as further analysable, so that each of its steps shall be atomic and each constant shall be introduced by its correspondent, "meaning-giving" rule. Thus, there would be again two – and again arguably significantly different – ways of expanding it in order to obtain such a goal:

$$\frac{\frac{\bot}{B}}{A \vee B} \quad \text{and} \quad \frac{\frac{\bot}{A}}{A \vee B}$$

This time, however, the divergent solutions – that similarly point at what can be regarded as an ambiguity – do not come about by virtue of any detours; all derivations involved are, in fact, normal.

For the cases considered, one could always claim, in the spirit of some category-theoretic considerations on the matter, that there should be no distinction between derivations of a given formula from absurdity, given that the latter is to be considered an *initial* object. Došen 2003, p.19, for instance, calls the denial of the initiality of absurdity "a desperate measure, not in tune with the other intuitions underlying the normalization conjecture" – but this might well be regarded as, at least, an overstatement. Another way of making the problem disappear is to restrict from the outset the application of absurdity rule to atomic conclusions only (see e.g. Prawitz 1971, pp. 242, 248) – but this seems, again, an *ad hoc* device, hardly in tune with the *ex falso quodlibet* understanding of absurdity. To advocate for the separation of the roles of logical constants, one should be expected to be able to *show* that absurdity can be restricted to atomic cases only; but this would rehabilitate the examples above involving disjunction.

This brief remark upon pressupositions of the normalisation thesis concerning how derivations work as representatives of proofs serves the purpose of exemplifying how these matters are far from inconsequential or uncontroverted, even in rather limited-scoped, formal treatments of identity of proofs such as the normalisation thesis. The possibility of association of more than one proof to a given representative due to ambiguity, in this case, poses a threat to the whole proposal. Arguing for the absence (or presence) of such phenomena is thus a basic requirement for attempts at dealing with the matter of when distinct proof (re)presentations represent the same proof(s) – clearly one of the most prominent questions in the agenda of the general proof theoretic treatment of identity of proofs.

b.2. More than one (re)presentation per proof

Just as it happens in the former case, it is also easy to devise contexts in which each proof has more than one (re)presentation. Think of e.g. presentations of Euclid's proof of the infinity of primes in textbooks, that differ in aspects such as notation, degree of formality, level of detail, definitions, etc. Or of how distinct natural deduction derivations may be looked upon as expressing the same argument or "proof idea" (cf. Prawitz 1971, p.257) – say, synonymous derivations in the sense of de Castro Alves 2019, pp.118-125.

b.3. Outset conditions, yet again

Just as in the case of section (a), we can talk about outset conditions to the investigation of identity of proofs formulated in terms of the quantitative relation between proofs and their (re)presentations just described. Again, the idea is that the decisions set "*a priori*" limits to how the matter of identity of proofs is to be treated. The questions at stake now are, first: Can a (re)presentation (re)present more than one proof (kind/collection)?; second: Can a proof(kind/collection) be (re)presented by more than one (re)presentation? Depending on how those are answered, the following outset conditions are established:

1) *A proof (kind/collection) can have more than one (re)presentation*: if this outset condition is in force, we call the context of discussion of identity of proofs *plural*. Otherwise, it is called *singular.*
2) *A (re)presentative can represent more than one proof (kind/collection):* if this outset condition is in force, we call the context of discussion of identity of proofs *indeterminate*. Otherwise, it is called *determinate*.

Let us now specify some terminology to facilitate the exposition:

3) *Synonymy:* Let π be a (re)presentation, σ be a (re)presentation and Π be a proof (kind/collection). We call π and σ synonymous with respect to Π iff both π and σ (re)present $\prod$ or neither π nor σ (re)present Π; we call π and σ simply synonymous iff π and σ are synonimous w.r.t. every Π.

4) *Univocity:* Let π be a (re)presentation. We call π univocal iff it (re)presents exactly one proof (kind/collection).

Now, depending on how the outset conditions just described are combined, different limits are imposed upon how proof (re)presentations can be said to be related to (the same) (kinds/collections of) proofs:

Can there be...			
Synonymy\ Univocity		..More than one proof per (re)presentation?	
		Yes	No
...More than one (re)presentation per proof?	Yes	Non-trivial Synonymy\ Non-trivial univocity	Non-trivial Synonymy\ trivial univocity
	No	Trivial synonymy\ Non-trivial univocity	1 to 1

As one can see, the decision between a plural or a singular context determines whether or not distinct proof representatives may be synonymous with respect to a proof (kind/collection) (and consequently also whether or not they may be synonymous *tout court*). The decision between an indeterminate or a determinate context, in turn, establishes whether or not the univocity of proof (re)presentations is guaranteed.

The point to be noted now is that the only combination that dispenses with the need of justification is "Yes – Yes": for it is the only one that does not block any possible rendering of the relation of synonymy and also does not decide whether or not univocity holds. Observations similar to the ones concerning the relation between proofs and proof results apply here as well: this variant does not mean that *in fact* there are distinct synonymous proof (re)presentations or that there are in fact non-univocal proof-(re)presentations; it only states that nothing of this much is assumed from the outset to hold with respect to identity of proofs. So, if we eventually come to the conclusion that e.g. (re)presentations (re)present only one proof each, this must not be an *a priori* limit to our investigation of equivalence criteria for proof (re)presentations, but should rather obtain, if at all, as an *outcome* of this investigation, and thus duly underpinned by proper arguments. All other possible combinations of outset conditions are in need of justification – i.e. quite unsurprisingly, one is expected to *show* why the possibility that e.g. one (re)presentation (re)presents multiple distinct proofs, or that no two distinct representations can be synonymous (w.r.t. some proof) should, if at all, be excluded from the outset.

c. Conclusion

We identified and described two different kinds of outset conditions that may be imposed upon the investigation of identity of proofs: one given in terms of how the identity of proofs is conditioned by the identity of what is proved (section a.), and other in terms of how equivalence relations between proof (re)presentations are conditioned by, on the one hand, how many distinct collections of proofs can be (re)presented by them, and, on the other, how many distinct (re)presentations a collection of proofs may have (section b.). These outset conditions allow one to devise some degree of cohesiveness or unity in the admittedly widely variegated talk of proofs in the literature, for they stem arguably innocently from what appear to be very generic traits of proofs, namely: proofs prove something, and proofs can be (re)presented. As long as a proof displays these features, the remarks made here apply to them somehow.

We also tried to illustrate how the thorough identification of this kind of outset condition, rather than consisting in negligible conceptual distinctions, has a decisive impact on how an influential formal account of identity of proofs in the literature – the normalisation thesis – should be evaluated as to the sufficiency of its philosophical underpinning. Given that the context of the observations made in this respect is rather restricted, not going beyond proofs in propositional logic, the importance of the remarks made becomes even more evident.

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